Chessboard-like Buckling Modes of Plates on Elastic Foundation

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Abstract. Due to the development of nano-technologies in electronic industry and bio-mechanics the raised interest appears to the problem of gaining the periodic structures on the surface at buckling in the micro-level. One of the simplest ways to solve this problem is to use the compressed plate on the soft elastic foundation [1-3]. The presented paper is devoted to the investigation of problem of gaining the chessboard-like buckling surface structures.

Introduction

The survey of some experimental works in which the chessboard-like buckling modes were observed one could find in [4]. It is shown that the critical initial deformation e_* and the corresponding surface wave length L at buckling may be calculated from the bifurcation equation, but the buckling mode remains unknown. To find the analytical solution for buckling mode it is necessary to study the initial supercritical behavior of the system. In [1, 3] it is shown that the mode

$$w(x,y) = w_0 \sin(px)\sin(qy) \tag{1}$$

with $p=q$ (the chessboard-like mode) corresponds to the minimum supercritical energy among all other modes of type (1). More buckling modes (including triangular and varicose modes) are studied in [5] and again the chessboard-like mode corresponds to the minimal energy. In the present paper we will establish that any function $w(x,y) = w_0\varphi(x,y)$, where $\varphi(x,y)$ satisfies the Helmholtz equation $\Delta\varphi + r^2\varphi = 0$, is the solution of the bifurcation equation. These functions wil be examined and it will be shown that the chessboard-like mode again gives the minimum energy.

We are going to study a plate consisiting of two-layered with similar mechanical properties on the soft elastic foundation taking into account two reasons of the appearance of initial stresses in the plate. The first one – is the difference in temperature extension between the foundation and the plate. Another one – prestresses in a thin upper layer of the nano-dimensional thickness grown on the lower plate layer [6]. If the equilibrium distances between atoms of crystal lattices of the layers are different then the initial stresses appear in the upper layer. In is known that, e.g. for atom lattices of silicon *Si* and germanium *Ge* the corresponding strain depends on the relative substance of *Ge* in the rigid mortar *SiGe* and could be up to 0.04.

In addition it will be demonstrated that the buckling amplitude could be controlled by the changing of the initial compression of a plate.

Bifurcation Equation and its Solutions

Let consider the stability loss of an uniformly compresses elastic plate on the soft elastic foundation. We will assume that the length L of the surface wave at buckling is essentially larger than the shell thickness h and smaller than the depth of foundation H (Fig. 1).

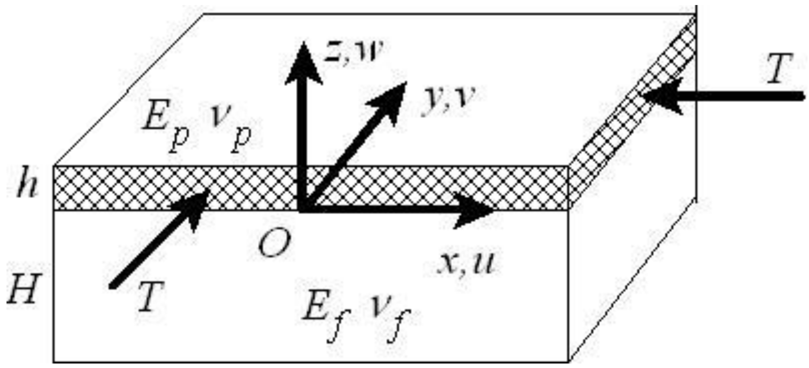

Fig. 1. Plate on an elastic foundation.

Then the buckling problem is reduced to the buckling of an infinite plate on the elastic half-space. This model is used in [2] where the metallic plate on the polymeric foundation is studied. Under this assumptions the Kirchhof-Love model could be accepted, and the critical compression T_* could be found from the bifurcation equation

$$D\Delta\Delta w - T\Delta w + P = 0, \quad \Delta Z = \frac{\partial^2 Z}{\partial x^2} + \frac{\partial^2 Z}{\partial y^2}. \tag{2}$$

Here $w(x,y)$ is the normal deflection; *x*, *y* are the Cartesian coordinates on the plate; *D* is the bending stiffness of the plate; Δ is the Laplace operator; *T* is a compression stress resultant in the plane of plate (*T>0* at compression); *P* is the normal stress between plate and foundation.
Supposing the full contact between the plate and the foundation we will take into account only the normal plate deflections. Naturally the stress $P(x,y)$ depends on the deflection $w(x,y)$. If $w(x,y) = w_0 \sin(px)\sin(qy)$, then [7,8]

$$P(x,y) = G_f r w(x,y), \quad r = \sqrt{p^2 + q^2}, \quad G_f = \frac{2E_f(1-\upsilon_f)}{(1+\upsilon_f)(3-4\upsilon_f)}, \tag{3}$$

where E_f, υ_f are the Young modulus and the Poisson ratio of the foundation. It is possible to prove that the same relation (3) is valid for any deflection $w(x,y)$ having the form $w(x,y) = w_0\varphi(x,y)$, where function $\varphi(x,y)$ satisfies the Helmholtz equation $\Delta\varphi + r^2\varphi = 0$.
Relations (2) and (3) yield

$$Dr^4 - Tr^2 + G_f r = 0, \tag{4}$$

whence it follows (after minimization by *r*) the expressions for the critical values r_* and $T_*$of the wave number *r* and the initial compression *T* [1,3]:

$$r_* = \left(\frac{G_f}{2D}\right)^{1/3}, \quad T_* = \frac{3G_f}{2r_*}. \tag{5}$$

Initial Super-Critical Deformation

Let seek the function $\varphi(x,y)$ which describes the buckling mode. Following [1] and [3] we will assume that $T > T_*$ and will try to investigate the initial super-critical behaviour of the system. The general form of the initial potential energy density *Q* is

$$Q(w_0, \delta, \varphi) = -\left(\frac{1}{2}\right) C_1(\varphi) T_* \delta r_*^2 w_0^2 + \left(\frac{1}{2}\right) C_2(\varphi) E_p h r_*^4 w_0^4 + O(w_0^5), \tag{6}$$

where $\delta = (T - T_*)/T_*$ is the relative growth of compression T compared with the critical value T_*, E_p is the (equivalent) Young modulus of plate and h is its thickness. The values $C_1(\varphi)$ and $C_2(\varphi)$ depend only on φ and υ_p and they equal to

$$\begin{gathered} C_1 = \langle \varphi^2 \rangle, \quad \langle Z \rangle = \frac{1}{S} \int_S Z dx dy, \\ C_2 = \frac{3 + \upsilon_p}{8(1 + \upsilon_p) r_*^4} \langle \left(\left(\frac{\partial \varphi}{\partial x} \right)^2 + \left(\frac{\partial \varphi}{\partial y} \right)^2 \right)^2 \rangle + \frac{5\upsilon_p - 1}{24(1 + \upsilon_p)} \langle \varphi^4 \rangle + \frac{\upsilon_p^2}{2(1 - \upsilon_p^2)} C_1^2. \end{gathered} \tag{7}$$

Here S is the cell of periodicity and brackets $\langle . \rangle$ denote averaging. The same relations (7) are valid for the almost periodic functions φ.

The first summand in (6) is obtained in [3]. To deliver the second summand one could calculate the density of plate extension-shear

$$\begin{gathered} Q_{ex} = \frac{E_p h}{2(1 - \upsilon_p^2) S} \int_S \left(\varepsilon_{xx}^2 + 2\upsilon_p \varepsilon_{xx} \varepsilon_{yy} + \varepsilon_{yy}^2 + \frac{(1 - \upsilon_p)\omega^2}{2} \right) dx dy, \\ \varepsilon_{xx} = \frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2, \quad \varepsilon_{yy} = \frac{\partial v}{\partial y} + \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2, \quad \omega = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y}, \end{gathered} \tag{8}$$

where $u(x, y)$ and $v(x, y)$ are the tangential deflections of the points belonging to the plate neutral plane. Here we neglect the energy of the non-linear deformation of foundation. Supposing that the function $w(x, y) = w_0 \varphi(x, y)$ is given we could find functions u, v imposing minimum of the functional Q_{ex} and then, after reduction, we could obtain the second summand in (6).

Criterion Functional

After minimization of (6) by w_0 the minimal value Q_* of the potential energy density and the corresponding value w_* of the deflection amplitude are expressed as

$$\frac{Q_*}{E_p h} = -\frac{k_T^2 \delta^2 \varepsilon^{4/3}}{4\Phi(\varphi)}, \quad \Phi = \frac{C_2}{C_1^2}, \quad \frac{w_*}{h} = c_w \delta^{1/2}, \quad c_w = \left(\frac{C_1 k_T}{2 C_2 k_r^2} \right)^{\frac{1}{2}}. \tag{9}$$

Let φ_1 and φ_2 be two solutions of the equation $\Delta\varphi + r^2\varphi = 0$. If $\Phi(\varphi_1) < \Phi(\varphi_2)$ then we may expect that the appearing buckling mode $w(x, y) = w_0 \varphi_1(x, y)$ is preferrable as compared with the mode $w(x, y) = w_0 \varphi_2(x, y)$. Referring $\Phi(\varphi)$ as the criterion functional and putting $C_1(\varphi) = 1$ without loss of generality one could conclude that $\Phi(\varphi) = C_2(\varphi)$.

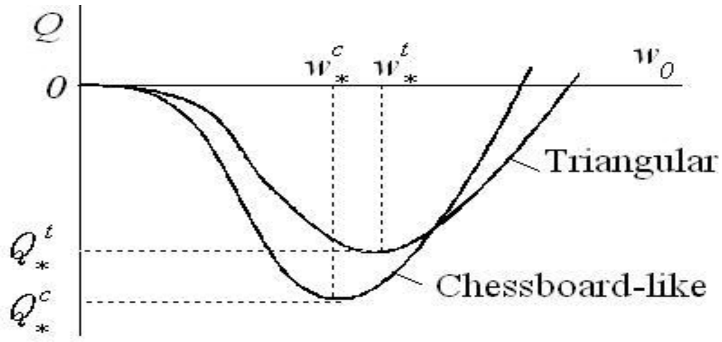

Fig. 2. Dependence $Q(w_0)$.

So, a number of functions $\varphi(x,y)$ satisfying the equation $\Delta\varphi + r^2\varphi = 0$ are examined. For the rectangular buckling mode $\varphi(x,y) = \sin(px)\sin(qy)$ equation (9) yields

$$\Phi = \frac{3-\upsilon_p^2}{4(1-\upsilon_p^2)} - \frac{(3+\upsilon_p)p^2q^2}{2(1+\upsilon_p)r_*^4} \tag{10}$$

and comparing the rest of the rectangles $\min\Phi = \Phi_* = (3-\upsilon_p)/8(1-\upsilon_p)$ is achieved for a square (p=q, Fig. 3*a*). For the right triangle (Fig. 3*b*) $\Phi = (11-5\upsilon_p)/\left(24(1-\upsilon_p)\right) > \Phi_*$.

The dependences $Q(w_0)$ given by (6) for two functions φ (chessboard-like and triangular) are shown in Fig. 2. It is clear that the equilibrium state $w = w_*$ is stable because in this point where the energy is minimal.

Some possible buckling modes (chessboard-like *(a)*, triangular *(b)*, rectangular *(c)*, and one-dimensional *(d)*) are shown at Fig. 3.

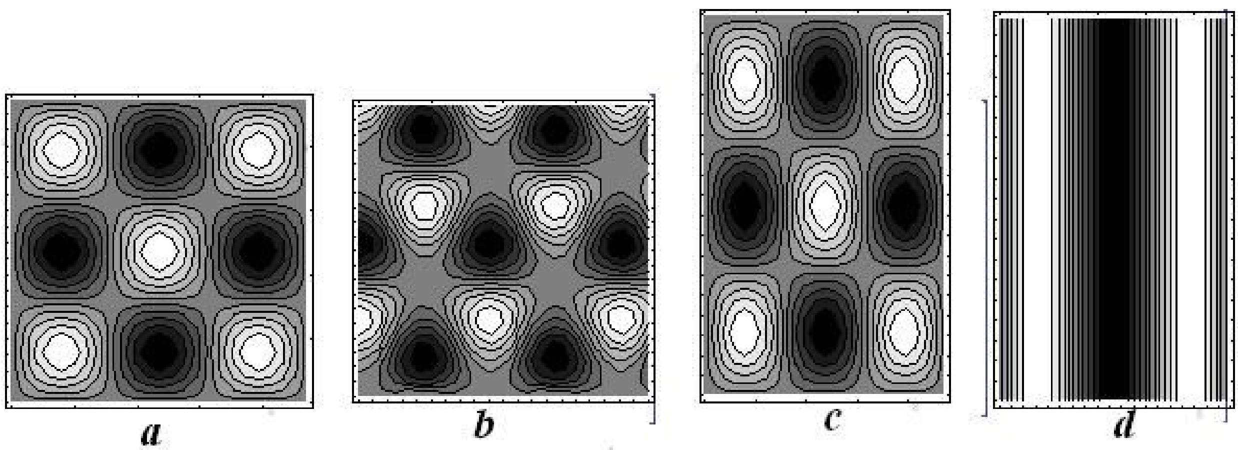

Fig. 3. Solutions of the bifurcation equation.

All the examined modes

$$\varphi(x,y) = \sum_k A_k \sin(p_k x + q_k y + \alpha_k), \quad p_k^2 + q_k^2 = r_*^2 \tag{11}$$

also lead to the inequality $\Phi > \Phi_*$. All these modes are the partial cases of relation (11). The corresponding values of the criterion functional are given in Table 1 for $\upsilon_p = 0.3$.

Table 1. Values of the criterion functional $\Phi(\varphi)$.

See Fig. 3	Buckling mode	$\varphi(x,y)$	$\Phi(\varphi)$
a	chessboard-like	$\sin x \sin y$	0.482
b	triangular	$\sin 2y \sin(y + x\sqrt{3}) \sin(y - x\sqrt{3})$	0.565
c	rectangular	$\sin 2x \sin y$	0.596
d	one-dimensional	$\sin x$	0.799

Therefore we may expect that the initial supercritical behavior of a compressed plate on the soft elastic foundation is a chessboard-like one.

Besires that, it follows from (9) that $w_* \sim \delta^{1/2} = (\frac{T - T_*}{T_*})^{1/2}$. That is, changing the compression T one can control the buckling amplitude w_*. In the next section we will study the dependence of critical compression on the temperature and on atomic structure of a two-layered plate.

Two-layered Compressed Plate on Elastic Foundation

Let consider the stability of a two-layered compressed plate lying on an elastic foundation (Fig. 4). We will denote the Young modules, the Poisson ratios, the thicknesses, the initial deformations and the coefficients of temperature extension of a foundation (*f*), and of a plates (*p1*, *p2*), respectively as

$$E_f, \upsilon_f, H, \varepsilon_f^0, \alpha_f; \qquad E_{pi}, \upsilon_{pi}, h_i, \varepsilon_{pi}^0, \alpha_{pi}, \quad i = 1,2 \quad h = h_1 + h_2 . \tag{12}$$

Here h is the plate thickness. Also we will assume for simplicity that $\upsilon_{p1} = \upsilon_{p2} = \upsilon_p$ and $\alpha_{p1} = \alpha_{p2} = \alpha_p$, and again, in order to use the Kirchhoff-Love model (1), that

$$E_f \ll \{E_{p1}, E_{p2}\}, \quad h \ll H. \tag{13}$$

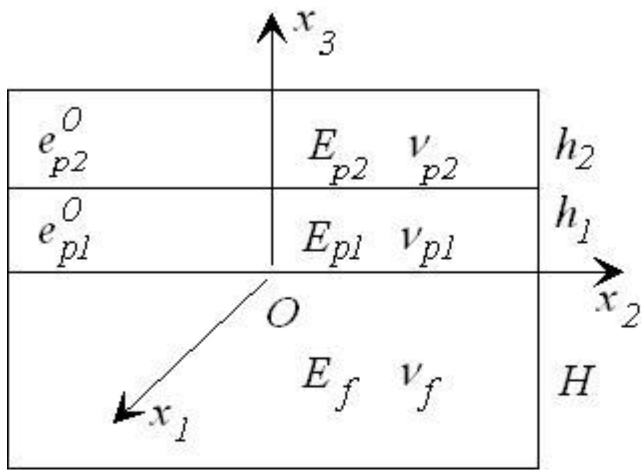

Fig. 4. Two-layered plate on elastic foundation.

For considered problem the parameters in (1) and (6) are given by

$$D = \frac{E_{p1}^2 h_1^4 + 2E_{p1}E_{p2}h_1h_2(2h_1^2 + 3h_1h_2 + 2h_2^2) + E_{p2}^2 h_2^4}{12(1 - \upsilon_p^2)(E_{p1}h_1 + E_{p2}h_2)}, \tag{14}$$

$$E_p h = E_{p1} h_1 + E_{p2} h_2$$

$$T = -\frac{E_{p1} h_1 \varepsilon_{p1}^0 + E_{p2} h_2 \varepsilon_{p2}^0}{1 - \upsilon_p}, \qquad \varepsilon_{p1}^0 = (\alpha_p - a_f)\Delta t, \quad \varepsilon_{p2}^0 = (\alpha_p - a_f)\Delta t + \varepsilon_\alpha,$$

where Δt is the changing of temperature, and ε_α is the initial deformation in the upper layer related to the difference of equilibrium distances between atoms of crystal lattices of layers. The deformation ε_α could appear only if the elastic properties of the first and the second layers are very similar.

To the approximate estimation of the parameters describing the critical situation let put $E_{p1} = E_{p2} = E_p$. Then formulas (5) and (14) yield the critical relation between parameters:

$$(\alpha_p - a_f)\Delta t + \frac{h_2}{h}\varepsilon_\alpha = c\left(\frac{E_f}{E_p}\right)^{2/3}, \qquad c = \frac{1}{(1+\upsilon_p)^{\frac{1}{3}}}\left(\frac{3(1-\upsilon_f)(1-\upsilon_p)}{2(1+\upsilon_f)(3-4\upsilon_f)}\right)^{\frac{2}{3}}. \qquad (15)$$

In order to receive evidence that the foundation is soft enough one could take, e.g. $\upsilon_f = \upsilon_p = 0.3$ (then c=0,423), $\alpha_p - a_f = 2.5 \cdot 10^{-5}$, $0 \le \Delta T \le 400°$, $\varepsilon_\alpha \le 0.04$, $\frac{h_2}{h} \le 0.5$ and then the relation (15) yields $E_f < 0.019\, E_p$.

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