An Intermediate Asymptotic Solution of the Coupled Creep-Damage Crack Problem in Similarity Variable

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Abstract. The concept of scaling and intermediate asymptotics is used for analysis of the creep-damage crack tip fields. It is shown that the intermediate asymptotics for stresses, strains and continuity parameter in the vicinity of the crack tip can be introduced. Using the intermediate variable the class of self-similar solutions to coupled (creep-damage) crack problems is solved. The constitutive model is based on continuum damage mechanics. The conventional Kachanov-Rabotnov creep-damage theory is utilized to study the asymptotic behavior of damage in the region very near the crack tip. The totally damaged zone where the damage (integrity) parameter reaches its critical value is assumed to exist in the vicinity of the crack tip. Using the similarity variable the asymptotic solutions to mode I, II and mode III crack problems are obtained. The asymptotic stress, creep strain rate and damage fields near the crack tip are analyzed by solving nonlinear eigenvalue problems resulting in a new far stress distribution. The configurations of the totally damaged zone governed by the new far stress field are found and analyzed.

Introduction

Analysis of effects of material damage on the stress and strain fields near crack tip in non-linear materials is the very important problem for evaluation of crack behavior in elements of structures. The influence of damage on crack-tip fields has been the subject of many papers, especially for cracks in brittle materials [1], elastic-plastic-brittle cracks [2, 3], creep cracks [4-6] and fatigue cracks [7]. Thus, the phenomenon of crack growth in materials undergoing deformations coupled with damage has been investigated extensively over the past twenty years. Some of the essential aspects of the considered set of two-dimensional crack problems and the results obtained can be highlighted. 1. The damage gives significant influence on the stress and strain (strain rate) fields near the crack tip. 2. The mathematical structure of governing equations is affected by the modelling of damage. 3. While the Hutchinson-Rice-Rosengren (HRR) – field of non-linear fracture mechanics always shows the stress singularity at the crack tip for any finite value of the stress exponent, the preceding material damage in front of the crack tip decreases the singularity, and may give non-singular stress field. 4. The totally damage and (or) active damage zone (process zone) need be modelled in the crack tip region.

In the present work the asymptotic stress, strain rate and continuity fields in the vicinity of mode I and mode III cracks in damaged materials are obtained using the self-similar variable proposed by Riedel [8]. The form of the similarity solution has been introduced by Riedel. However there exist no solutions where the similarity property of damage mechanics equations is used. The advantage of a similarity solution is that it reduces the number of independent variables in the problem by one. This simplification allows us to gain insight into the time evolution of the near tip stress fields and the far field boundary condition.

In discussing crack growth on the basis of damage mechanics it is advantageous to introduce the term "totally damaged zone". The totally damaged zone can be interpreted as a zone occupied by microcracks oriented orthogonally to the main crack. Inside the totally damaged zone (TDZ) the damage involved reaches its critical value (for instance, the damage parameter reaches unity) and a complete fracture failure occurs. In view of material damage stresses are relaxed to vanishing. Therefore one can assume that the stress tensor components in the TDZ equal zero. Outside the zone damage alters the stress distribution substantially compared to the corresponding non-damaging material. Well outside that zone the damage parameter is equal to 1.

In the present study mode I, II and III crack problems for power-law creeping materials are considered by employing the self-similar variable on the assumption that the TDZ in the vicinity of the crack tip exists.

Self-similar variable and self-similar representation of the solution. A static mode I crack problem in a damaged creeping material under the plane strain and plane stress conditions is considered. The equilibrium and compatibility equations in the polar coordinate system can, respectively, be written as

$$\frac{\partial\sigma_{rr}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{r\theta}}{\partial\theta}+\frac{\sigma_{rr}-\sigma_{\theta\theta}}{r}=0,\quad \frac{\partial\sigma_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta}+2\frac{\sigma_{r\theta}}{r}=0, \tag{1}$$

$$2\frac{\partial}{\partial r}\left(r\frac{\partial\dot{\varepsilon}_{r\theta}}{\partial\theta}\right)=\frac{\partial^2\dot{\varepsilon}_{rr}}{\partial\theta^2}-r\frac{\partial\dot{\varepsilon}_{rr}}{\partial r}+r\frac{\partial^2\left(r\dot{\varepsilon}_{\theta\theta}\right)}{\partial r^2}.$$

The creep power-law constitutive equations in the coupled creep-damage formulation are described by

$$\dot{\varepsilon}_{rr}=-\dot{\varepsilon}_{\theta\theta}=\frac{3B}{4}\left(\frac{\sigma_e}{\psi}\right)^{n-1}\frac{\sigma_{rr}-\sigma_{\theta\theta}}{\psi},\quad \dot{\varepsilon}_{r\theta}=\frac{3B}{2}\left(\frac{\sigma_e}{\psi}\right)^{n-1}\frac{\sigma_{r\theta}}{\psi}, \tag{2}$$

where $\sigma_e^2=3\left(\sigma_{rr}-\sigma_{\theta\theta}\right)^2/4+3\sigma_{r\theta}^2$ for plane strain conditions,

$$\dot{\varepsilon}_{rr}=\frac{B}{2}\left(\frac{\sigma_e}{\psi}\right)^{n-1}\frac{2\sigma_{rr}-\sigma_{\theta\theta}}{\psi},\quad \dot{\varepsilon}_{\theta\theta}=\frac{B}{2}\left(\frac{\sigma_e}{\psi}\right)^{n-1}\frac{2\sigma_{\theta\theta}-\sigma_{rr}}{\psi},\quad \dot{\varepsilon}_{r\theta}=\frac{3B}{2}\left(\frac{\sigma_e}{\psi}\right)^{n-1}\frac{\sigma_{r\theta}}{\psi} \tag{3}$$

where $\sigma_e^2=\sigma_{rr}^2+\sigma_{\theta\theta}^2-\sigma_{rr}\sigma_{\theta\theta}+3\sigma_{r\theta}^2$ for plane stress conditions.

The damage evolves according to

$$d\psi/dt=-A\left(\sigma_{eqv}/\psi\right)^m. \tag{4}$$

The traction-free conditions on the crack surfaces yield

$$\sigma_{\theta\theta}(r,\theta=\pm\pi)=0,\quad \sigma_{r\theta}(r,\theta=\pm\pi)=0. \tag{5}$$

The remote boundary condition has the form

$$\sigma_{ij}(r\to\infty,\theta,t)\to\left(C^*/\left(BI_n r\right)\right)^{-1/(n+1)}\tilde{\sigma}_{ij}(\theta,n). \tag{6}$$

The dimensionless constant I_n and the θ-variation functions of the suitably normalized functions $\tilde{\sigma}_{ij}(\theta,n)$ depend only on the creep exponent n.

If damage develops in the region which is small compared to the creep zone (the small scale damage conception), the boundary conditions (6) require that the stress field must approach the Hutchinson-Rice-Rosengren (HRR) field at large distances from the crack tip.

Dimensional analysis of Eqs. (1) – (6) shows that the damage mechanics equations must have similarity solutions of the form [8]

$$\sigma_{ij}(r,\theta,t)=\left(At\right)^{-1/m}\Sigma_{ij}(R,\theta),\quad \psi(r,\theta,t)=\Psi(R,\theta), \tag{7}$$

with the similarity variable

$$R=r\left(At\right)^{-(n+1)/m}BI_n/C^*.$$

The dimensionless functions $\Sigma_{ij}(R,\theta)$ and $\Psi(R,\theta)$ are as yet unknown. The validity of these similarity presentations of the solutions should be verified by insertion of (7) into governing equations and boundary conditions. For a more general remote boundary condition

$$\sigma_{ij}(r\to\infty,\theta,t)\to\tilde{C}r^s\tilde{\sigma}_{ij}(\theta,n)$$

one can introduce the similarity variable

$$R=r\left[tA\tilde{C}^m\right]^{1/(sm)}$$

This self – similar variable introduced is the self-similar variable of the second kind (incomplete similarity). The self-similar variable and the ordinary differential equations following from the continuum and damage mechanics equations have the unknown parameter s. For arbitrary values of this parameter the solution of the equations does not exist. The equilibrium and compatibility equations hold their forms while the kinetics evolution law of damage takes the following form

$$R\partial\psi/\partial R=-sm\left(\Sigma_{eqv}/\Psi\right)^m, \qquad (8)$$

where $\Sigma_{eqv}=\alpha\Sigma_1+\beta\Sigma_e+(1-\alpha-\beta)\Sigma$ is the damage equivalent stress in terms of the similarity variable.

The Airy stress potential $F(r,\theta)$ can be used to obtain

$$\sigma_{\theta\theta}=F,_{rr}\quad \sigma_{rr}=\Delta F-\sigma_{\theta\theta}\quad \sigma_{r\theta}=-\left(r^{-1}F,_{\theta}\right),_r$$

It is assumed that the Airy stress function and the integrity (continuity) parameter at large distances from the crack tip $(R\to\infty)$ are separable and can be expressed as series as

$$F(R,\theta)=R^{\lambda+1}f^{(0)}(\theta)+R^{\lambda_1+1}f^{(1)}(\theta)+R^{\lambda_2+1}f^{(2)}(\theta)+R^{\lambda_3+1}f^{(3)}(\theta)+o\left(R^{\lambda_3+1}\right) \qquad (9)$$

$$\Psi(R,\theta)=1-R^{\gamma_1}g^{(1)}(\theta)-R^{\gamma_2}g^{(2)}(\theta)-R^{\gamma_3}g^{(3)}(\theta)-R^{\gamma_4}g^{(4)}(\theta)+o\left(R^{\gamma_5}\right) \qquad (10)$$

$\lambda>\lambda_1>\lambda_2>\lambda_3>\ldots$. The multi-term asymptotic stress expansions can be written in the form

$$\sigma_{RR}(R,\theta)=R^s\left[\lambda f^{(0)}+\left(f^{(0)}\right)''\right]+R^{s_1}\left[\lambda_1 f^{(1)}+\left(f^{(1)}\right)''\right]+R^{s_2}\left[\lambda_2 f^{(2)}+\left(f^{(2)}\right)''\right]+R^{s_3}\left[\lambda_3 f^{(3)}+\left(f^{(3)}\right)''\right]+\ldots,$$

$$\sigma_{\theta\theta}(R,\theta)=R^s\lambda(\lambda-1)f^{(0)}+R^{s_1}\lambda_1(\lambda_1-1)f^{(1)}+R^{s_2}\lambda_2(\lambda_2-1)f^{(2)}+R^{s_3}\lambda_3(\lambda_3-1)f^{(3)}+\ldots,$$

$$\sigma_{r\theta}(R,\theta)=R^s(1-\lambda)\left(f^{(0)}\right)'+R^{s_1}(1-\lambda_1)\left(f^{(1)}\right)'+R^{s_2}(1-\lambda_2)\left(f^{(2)}\right)'+R^{s_3}(1-\lambda_3)\left(f^{(3)}\right)'+\ldots$$

Where $s=\lambda-1, s_1=\lambda_1-1, s_2=\lambda_2-1, s_3=\lambda_3-1$.

The three-term asymptotic creep strain rate expansions as $R\to\infty$ for plane strain conditions are determined by the formulae

$$\dot{\varepsilon}_{RR}(R,\theta)=-\dot{\varepsilon}_{\theta\theta}(R,\theta)=R^{sn}\varepsilon_{RR}^{(0)}(\theta)+R^{s(n+m)}\varepsilon_{RR}^{(1)}(\theta)+R^{s(n+2m)}\varepsilon_{RR}^{(2)}(\theta)+R^{s(n+3m)}\varepsilon_{RR}^{(3)}(\theta)+\ldots \qquad (11)$$

$$\dot{\varepsilon}_{R\theta}(R,\theta)=R^{sn}\varepsilon_{R\theta}^{(0)}(\theta)+R^{s(n+m)}\varepsilon_{R\theta}^{(1)}(\theta)+R^{s(n+2m)}\varepsilon_{R\theta}^{(2)}(\theta)+R^{s(n+3m)}\varepsilon_{R\theta}^{(3)}(\theta)+\ldots \qquad (12)$$

where

$$\varepsilon_{RR}^{(0)}(\theta)=f_e^{n-1}\left[\left(1-\lambda^2\right)f^{(0)}+\left(f^{(0)}\right)''\right],\quad \varepsilon_{R\theta}^{(0)}(\theta)=-f_e^{n-1}\lambda\left(f^{(0)}\right)',$$

$$\varepsilon_{RR}^{(1)}=\frac{1}{2}f_e^{n-1}\left\{\left[\left(1-\lambda_1^2\right)f^{(1)}+\left(f^{(1)}\right)''\right]+\left[\left(1-\lambda^2\right)f^{(0)}+\left(f^{(0)}\right)''\right]\left[(n-1)f_e^{(1)}+ng^{(0)}\right]\right\}$$

$$\varepsilon_{R\theta}^{(1)}=-f_e^{n-1}\left\{\lambda_1\left(f^{(1)}\right)'+\lambda\left(f^{(0)}\right)'\left[(n-1)f_e^{(1)}+ng^{(0)}\right]\right\}$$

$$\varepsilon_{RR}^{(2)}=\frac{1}{2}f_e^{n-1}\left\{\left[\left(1-\lambda_2^2\right)f^{(2)}+\left(f^{(2)}\right)''\right]+\left[\left(1-\lambda_1^2\right)f^{(1)}+\left(f^{(1)}\right)''\right]\left[(n-1)f_e^{(1)}+ng^{(0)}\right]+\right.$$

$$\left.+\left[\left(1-\lambda^2\right)f^{(0)}+\left(f^{(0)}\right)''\right]\left[n\left(\frac{n+1}{2}\left(g^{(0)}\right)^2+g^{(1)}+(n-1)g^{(0)}f_e^{(1)}\right)+\frac{n-1}{2}\left((n-2)f_e^{(1)2}+2F_e^2\right)\right]\right\}$$

$$\varepsilon_{R\theta}^{(2)} = -f_e^{\,n-1}\Big\{\lambda_2\left(f^{(2)}\right)' + \lambda_1\left(f^{(1)}\right)'\left[(n-1)f_e^{(1)} + ng^{(0)}\right] +$$
$$+\lambda\left(f^{(0)}\right)'\left[n\left(\frac{n+1}{2}\left(g^{(0)}\right)^2 + g^{(1)} + (n-1)g^{(0)}f_e^{(1)}\right) + \frac{n-1}{2}\left((n-2)f_e^{(1)2} + 2F_e^2\right)\right]\Big\}$$
$$f_e = \sqrt{\left[\left(1-\lambda^2\right)f^{(0)} + \left(f^{(0)}\right)''\right] + 4\lambda^2\left(f^{(0)}\right)'^2},\ F_e^2 = \frac{1}{2}\left[f_e^{(2)} - \left(f_e^{(1)}\right)^2\right]$$
$$f_e^{(1)} = \left\{\left[\left(1-\lambda^2\right)f^{(0)} + \left(f^{(0)}\right)''\right]\left[\left(1-\lambda_1^2\right)f^{(1)} + \left(f^{(1)}\right)''\right] + 4\lambda\lambda_1\left(f^{(0)}\right)'\left(f^{(1)}\right)'\right\}f_e^{-2},$$
$$f_e^{(2)} = \left\{\left[\left(1-\lambda_1^2\right)f^{(1)} + \left(f^{(1)}\right)''\right]^2 + 4\lambda_1^2\left[\left(f^{(1)}\right)'\right]^2 +\right.$$
$$\left.+\left[\left(1-\lambda^2\right)f^{(0)} + \left(f^{(0)}\right)''\right]\left[\left(1-\lambda_2^2\right)f^{(2)} + \left(f^{(2)}\right)''\right] + 8\lambda\lambda_2\left(f^{(0)}\right)'\left(f^{(2)}\right)'\right\}f_e^{-2}.$$

Using the asymptotic expansions (9), (10) and the compatibility equation (2) one finds

$$2(sn+1)\varepsilon_{R\theta,\theta}^{(0)} = \varepsilon_{RR,\theta\theta}^{(0)} - sn(sn+2)\varepsilon_{RR}^{(0)}, \tag{13}$$
$$2[(sn+m)+1]\varepsilon_{R\theta,\theta}^{(1)} = \varepsilon_{RR,\theta\theta}^{(1)} - s(n+m)[s(n+m)+2]\varepsilon_{RR}^{(1)},$$
$$2[(sn+2m)+1]\varepsilon_{R\theta,\theta}^{(2)} = \varepsilon_{RR,\theta\theta}^{(2)} - s(n+m)[s(n+2m)+2]\varepsilon_{RR}^{(2)}.$$

Taking into account Eqs. (13) one can obtain the nonlinear ordinary differential equation with respect to $f^{(0)}(\theta)$:

$$f_e^2\left(f^{(0)}\right)^{IV}\left\{(n-1)\left[\left(1-\lambda^2\right)f^{(0)} + \left(f^{(0)}\right)''\right]^2 + f_e^2\right\} + (n-1)(n-3)\times$$
$$\times\left\{\left[\left(1-\lambda^2\right)f^{(0)} + \left(f^{(0)}\right)''\right]\left[\left(1-\lambda^2\right)\left(f^{(0)}\right)' + \left(f^{(0)}\right)'''\right] + 4\lambda^2\left(f^{(0)}\right)'\left(f^{(0)}\right)''\right\}^2\left[\left(1-\lambda^2\right)f^{(0)} + \left(f^{(0)}\right)''\right] +$$
$$+(n-1)f_e^2\left\{\left[\left(1-\lambda^2\right)\left(f^{(0)}\right)' + \left(f^{(0)}\right)'''\right]^2 + \left[\left(1-\lambda^2\right)f^{(0)} + \left(f^{(0)}\right)''\right]\left(1-\lambda^2\right)\left(f^{(0)}\right)'' +\right.$$
$$\left.+4\lambda^2\left(f^{(0)''2} + f^{(0)'}f^{(0)'''}\right)\right\}\left[\left(1-\lambda^2\right)f^{(0)} + \left(f^{(0)}\right)''\right] + 2(n-1)f_e^2\times \tag{14}$$
$$\times\left\{\left[\left(1-\lambda^2\right)f^{(0)} + \left(f^{(0)}\right)''\right]\left[\left(1-\lambda^2\right)\left(f^{(0)}\right)' + \left(f^{(0)}\right)'''\right] + 4\lambda^2\left(f^{(0)}\right)'\left(f^{(0)}\right)''\right\}\left[\left(1-\lambda^2\right)\left(f^{(0)}\right)' + \left(f^{(0)}\right)'''\right] +$$
$$+C_1(n-1)f_e^2\left\{\left[\left(1-\lambda^2\right)f^{(0)} + \left(f^{(0)}\right)''\right]\left[\left(1-\lambda^2\right)\left(f^{(0)}\right)' + \left(f^{(0)}\right)'''\right] + 4\lambda^2\left(f^{(0)}\right)'\left(f^{(0)}\right)''\right\}\left(f^{(0)}\right)' +$$
$$+C_1 f_e^4\left(f^{(0)}\right)'' - C_2 f_e^4\left[\left(1-\lambda^2\right)f^{(0)} + \left(f^{(0)}\right)''\right] + f_e^4\left(1-\lambda^2\right)\left(f^{(0)}\right)'' = 0,$$

where, for brevity's sake, the following notations are adopted

$$C_1 = 4\lambda[(\lambda-1)n+1], \quad C_2 = (\lambda-1)n[(\lambda-1)n+2].$$

The fourth order nonlinear ordinary differential equation (14) with the boundary conditions

$$f^{(0)}(\theta = \pm\pi) = 0, \quad \left(f^{(0)}\right)'(\theta = \pm\pi) = 0 \tag{15}$$

defines a nonlinear eigenvalue problem in which the constant λ is the eigenvalue and $f^{(0)}(\theta)$ the corresponding eigenfunction.

Nonlinear eigenvalue problem. The problem of stress singularity is reduced to a nonlinear eigenvalue problem. Nowadays the whole eigenspectrum and orders of stress singularity at the crack tip for a power-law medium are of prevailing interest. The whole eigenspectrum stipulates the possible stress distributions in the neighborhood of the crack tip in a damaged medium. The shooting procedure commonly employed for nonlinear eigenvalue problems becomes multi-parametric for mode I and II crack problems and the numerical results obtained need to be proved additionally. To overcome this difficulty in the problem the perturbation theory approach has been applied. A further reason to consider this problem is in the need for a formula expressing eigenvalues for the nonlinear problem through eigenvalues of the linear problem and the creep exponent. The method based on the perturbation theory proposed in [9] and developed in [10] allows us to find the whole eigenspectrum and orders of stress field at the crack tip for a power-law medium.

The underlying idea of the method is to consider the expansion representing the eigenvalue λ of the nonlinear eigenvalue problem (14), (15) for an arbitrary exponent n to be a sum of the eigenvalue λ_0 corresponding to the "undisturbed" linear problem $(n=1)$ and a small parameter ε which quantitatively describes the nearness of the eigenvalues:

$$\lambda = \lambda_0 + \varepsilon . \quad (16)$$

The creep exponent n and the stress function $f^{(0)}(\theta)$ can be presented as formal series with respect to ε

$$n = 1 + \varepsilon n_1 + \varepsilon^2 n_2 + \varepsilon^3 n_3 + ..., \quad f^{(0)}(\theta) = f_0(\theta) + \varepsilon f_1(\theta) + \varepsilon^2 f_2(\theta) + \varepsilon^3 f_3(\theta) + ... \quad (17)$$

where $f_0(\theta)$ denotes the solution of the linear problem $(n=1)$. Introducing (16), (17) into (14) and collecting terms of equal power in ε, the set of linear differential equations is obtained. The first equation describing the linear problem has the following solution

$$f_0(\theta) = B_1 \cos[(\lambda_0 - 1)\theta] + B_2 \sin[(\lambda_0 - 1)\theta] + B_3 \cos[(\lambda_0 + 1)\theta] + B_4 \sin[(\lambda_0 + 1)\theta].$$

The boundary conditions lead to the well known characteristic equation $\sin 2\lambda_0 \pi = 0$. As is expected, the eigenspectrum of the linear problem distributes discretely and has infinite number of eigenvalues $\lambda_0 = m/2$. Considering mode I crack problem and odd integers m here one can represent the solution of the linear problem in the form

$$f_0(\theta) = \beta \cos(\alpha\theta) - \alpha \cos(\beta\theta), \quad \alpha = \lambda_0 - 1, \beta = \lambda_0 + 1.$$

The dimensionless angular function $f_1(\theta)$ must satisfy the fourth order linear ordinary differential equation

$$f_1^{IV} + 2(\lambda_0^2 + 1) f_1'' + (\lambda_0^2 - 1)^2 f_1 = -n_1 \frac{x_0 \left(f_0^{IV} x_0 + w_0 \right)}{g_0} + 2\lambda_0 f_0'' - C_1^1 f_0'' + C_2^1 x_0 + 2\lambda_0 a_0 f_0 , \quad (18)$$

where the following notations are accepted

$$a_0 = 1 - \lambda_0^2, \quad x_0 = a_0 f_0 + f_0'', \quad g_0 = x_0^2 + 4\lambda_0^2 (f_0')^2$$

$$w_0 = (x_0')^2 + a_0 x_0 f_0'' + 4\lambda_0^2 (f_0'')^2 + 4\lambda_0^2 f_0' f_0'',$$

$$C_1^1 = 4\lambda_0 [2 + n_1(\lambda_0 - 1)], \quad C_2^1 = 2\lambda_0 [1 + n_1(\lambda_0 - 1)].$$

Boundary conditions follow from the traction free conditions on the crack faces:

$$f_1(\theta = \pm\pi) = 0, \quad f_1'(\theta = \pm\pi) = 0 . \quad (19)$$

Thus, the two-point boundary value problem (18), (19) for nonhomogeneous fourth order linear differential equation is formulated. It is known that if the boundary value problem the homogeneous differential equation has a nontrivial solution then there can exist no solution of the corresponding nonhomogeneous differential equation unless the solvability condition is realized.

The solvability condition can be formulated by using a solution of the self-adjoint problem

$$\int_{-\pi}^{\pi} u g(\theta) d\theta = 0, \quad u = f_0(\theta) = \beta \cos(\alpha\theta) - \alpha \cos(\beta\theta) \quad (20)$$

where u is the solution of the self-adjoint problem corresponding to (18), (19), $g(\theta)$ is the right side of (18). The solvability condition (20) enables to obtain the first perturbation of n:

$$n_1 = -\frac{2}{\lambda_0 - 1},$$

and, consequently, the two-term asymptotic expansion for the exponent n has the following form

$$n = 1 - \frac{2\varepsilon}{\lambda_0 - 1} + O(\varepsilon^2) .$$

The nonhomogeneous linear differential equation for the function $f_2(\theta)$ can be presented as

$$\begin{aligned}
& g_0^2 \left[f_2^{IV} + 2\left(\lambda_0^2+1\right) f_2'' + \left(\lambda_0^2-1\right)^2 f_2 \right] + \\
& + g_0^2 \left(-x_0 + C_1^2 f_0'' - C_2^2 x_0 + 2\lambda_0 C_2^1 f_0 \right) + n_1 \left\{ -x_0 \left(f_0^{IV} x_0 + w_0 \right) \left[-4\lambda_0 f_0 x_0 + 8\lambda_0 \left(f_0' \right)^2 \right] + \right. \\
& + g_0 x_0 \left[-4\lambda_0 x_0' f_0' - 2\lambda_0 a_0 f_0 f_0'' - 2\lambda_0 x_0 f_0'' + 8\lambda_0 \left(f_0'' \right)^2 + 8\lambda_0 f_0' f_0''' \right] + \\
& + 2h_0 x_0' \left[-4\lambda_0 x_0 f_0 + 8\lambda_0 \left(f_0' \right)^2 \right] - 2h_0 x_0 \left[-2\lambda_0 x_0 f_0' - 2\lambda_0 x_0' f_0 + 8\lambda_0 f_0' f_0'' \right] + \\
& + 4\lambda_0^2 h_0 f_0' \left[-4\lambda_0 x_0 f_0 + 8\lambda_0 \left(f_0' \right)^2 \right] - 2\lambda_0 g_0 f_0 \left(f_0^{IV} x_0 + w_0 \right) - 2\lambda_0 g_0 f_0 f_0^{IV} x_0 + n_1 h_0^2 x_0 + \qquad (21) \\
& + 4\lambda_0 h_0^2 f_0 - 4\lambda_0 g_0 h_0 f_0' - x_0 \left(f_0^{IV} x_0 + w_0 \right) \left[2x_0 x_1 + 8\lambda_0^2 f_0' f_1' \right] + g_0 x_0^2 f_1^{IV} + \\
& + g_0 x_0 \left[2x_0' x_1' + a_0 f_0'' x_1 + a_0 x_0 f_1'' + 8\lambda_0^2 f_0'' f_1'' + 4\lambda_0^2 f_0''' f_1' + 4\lambda_0^2 f_0' f_1''' \right] + 2h_0 x_0' \left[2x_0 x_1 + 8\lambda_0^2 f_0' f_1' \right] - \\
& - 2h_0 x_0 \left[x_0' x_1 + x_0 x_1' + 4\lambda_0^2 f_0' f_1'' + 4\lambda_0^2 f_0'' f_1' \right] + 4\lambda_0^2 h_0 f_0' \left[2x_0 x_1 + 8\lambda_0^2 f_0' f_1' \right] + \\
& \left. + g_0 \left(f_0^{IV} x_0 + w_0 \right) x_1 + 2h_0 g_0 x_1' - 2h_0^2 x_1 + 4\lambda_0^2 h_0 g_0 f_1' + f_0^{IV} g_0 x_0 x_1 \right\} = 0,
\end{aligned}$$

where $h_0 = x_0 x_0' + 4\lambda_0^2 f_0' f_0''$, $\quad x_1 = a_0 f_1 + f_1''$,

$C_1^2 = 4\left\{ \lambda_0 \left[n_1 + n_2 \left(\lambda_0 - 1 \right) \right] + 1 + n_1 \left(\lambda_0 - 1 \right) \right\}$, $\quad C_2^2 = 2\lambda_0 \left[n_1 + n_2 \left(\lambda_0 - 1 \right) \right] + \left[1 + n_1 \left(\lambda_0 - 1 \right) \right]^2$.

The solvability condition of the boundary value problem for Eq. (21) allows to find

$$n_2 = -\frac{\lambda_0^5 - 2\lambda_0^4 - 7\lambda_0^3 + 11\lambda_0^2 + 4\lambda_0 - 5 - \left(\lambda_0^2 - 1\right) \operatorname{sgn}(\lambda_0)}{\left(\lambda_0 + 1\right)\left(\lambda_0 - 1\right)^4} .$$

Thus the three-term asymptotic expansion for the creep exponent has the form

$$n = 1 - \frac{2\varepsilon}{\lambda_0 - 1} + \varepsilon^2 n_2(\lambda_0) + O(\varepsilon^3) .$$

For some eigenvalues of the linear problem λ_0 one can obtain the four-term asymptotic expansions:

$$\lambda_0 = -1/2 \qquad n = 1 + \frac{4}{3}\varepsilon + \frac{92}{81}\varepsilon^2 + \frac{2576}{2187}\varepsilon^3 + O(\varepsilon^4)$$

$$\lambda_0 = 1/2 \qquad n = 1 + 4\varepsilon + 8\varepsilon^2 + 16\varepsilon^3 + O(\varepsilon^4)$$

$$\lambda_0 = 3/2 \qquad n = 1 - 4\varepsilon + \frac{53}{5}\varepsilon^2 + \frac{4531}{50}\varepsilon^3 + O(\varepsilon^4)$$

$$\lambda_0 = 5/2 \qquad n = 1 - \frac{4}{3}\varepsilon + \frac{683}{567}\varepsilon^2 + \frac{199043}{214326}\varepsilon^3 + O(\varepsilon^4)$$

$$\lambda_0 = 7/2 \qquad n = 1 - \frac{4}{5}\varepsilon - \frac{613}{1875}\varepsilon^2 - \frac{817069}{1406250}\varepsilon^3 + O(\varepsilon^4)$$

Thus, using the perturbation method the whole set of eigenvalues for mode I and mode II crack problem for a power-law creeping material is determined. The three-term (or four-term) asymptotic expansions for the creep exponent allowing to find the eigenvalue via Eq. (16) for the nonlinear eigenvalue problem are obtained.

Geometry of the totally damaged zone (TDZ). The eigenvalues resulting in the contours of the TDZ converging to the limit contour have been found. It turns out that the HRR stress field does not govern the geometry of the TDZ. The new intermediate stress asymptotic is obtained. The shapes of the TDZ obtained for the new stress asymptotics at large distances from the crack tip for plane strain and plane stress conditions are shown in Fig. 1, 2, where 1 – the contour given by the two-term asymptotic expansion of the integrity parameter, 2 – the contour given by the three-term asymptotic expansion of the integrity parameter, 3 – the contour given by the four-term asymptotic expansion of the integrity parameter.

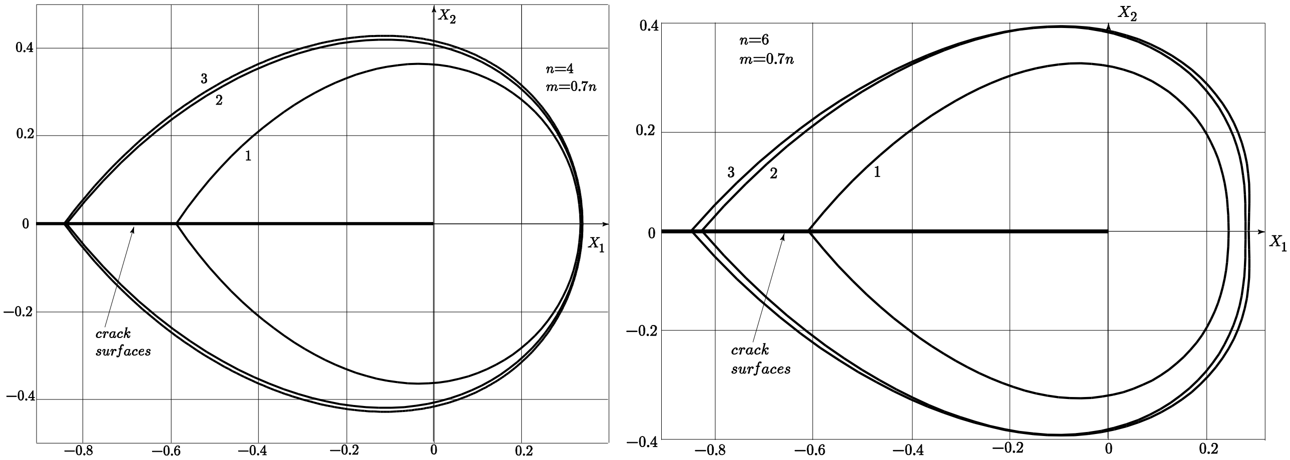

Fig. 1. The boundary of the TDZ for different values of material constants (plane strain conditions)

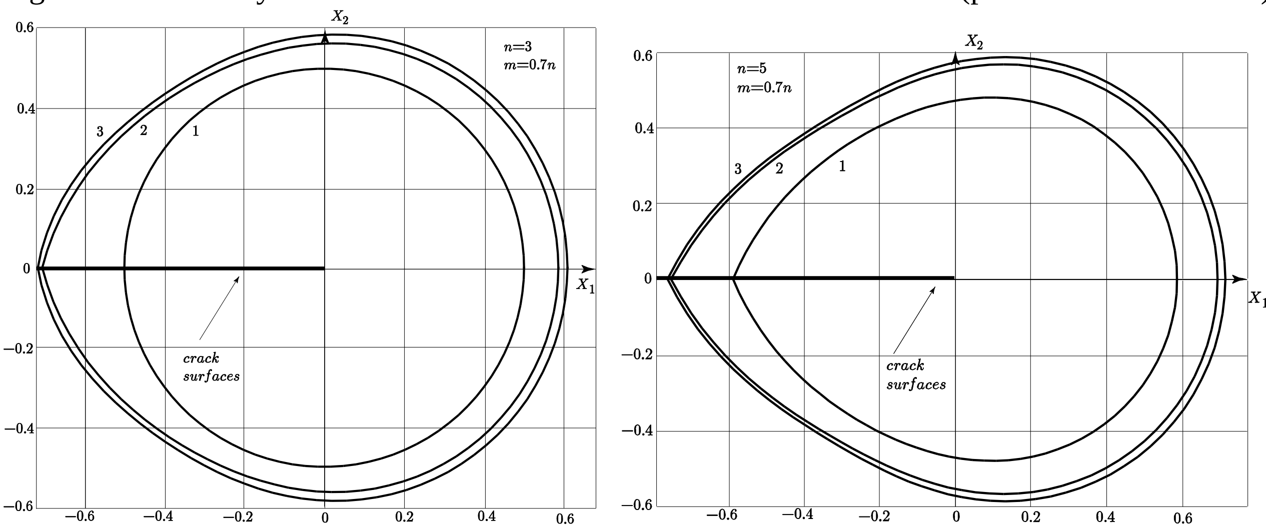

Fig. 2. The boundary of the TDZ for different values of material constants (plane stress conditions)

Finite difference method solution. To justify the asymptotic solution obtained one can address to the direct numerical integration of equations formulated in terms of the similarity variable. The numerical solution has been found by the finite difference method. The numerical solution achieved exhibits the same characteristic features of the self-similar ansatz revealed by the approximate approach: that is, the intermediate asymptotic behaviour of the stresses at distances considerably beyond the TDZ length but at yet still small distances comparatively with the crack length occurs. It is interesting to represent the effective stress in double logarithmic coordinates (Fig. 3). It is seen that there are two rectilinear parts: one linear region corresponds to the HRR-filed while the order linear part corresponds to the new intermediate asymptotic solution. The intermediate asymptotic solution is the stress and integrity distributions valid for times and distances at which the influence of fine details of initial and boundary conditions is lost [11-14].

Summary

The class of self-similar solutions to coupled (creep-damage) crack problems is presented. The constitutive model is based on continuum damage mechanics. The conventional Kachanov-Rabotnov creep-damage theory is utilized to study the asymptotic behavior of damage in the region very near the crack tip. The totally damaged zone where the damage (integrity) parameter reaches its critical value is assumed to exist in the vicinity of the crack tip. Using the similarity variable the asymptotic solutions to mode I, II and mode III crack problems are obtained. The asymptotic stress, creep strain rate and damage fields near the crack tip are analyzed by solving nonlinear eigenvalue problems resulting in a new far stress distribution.

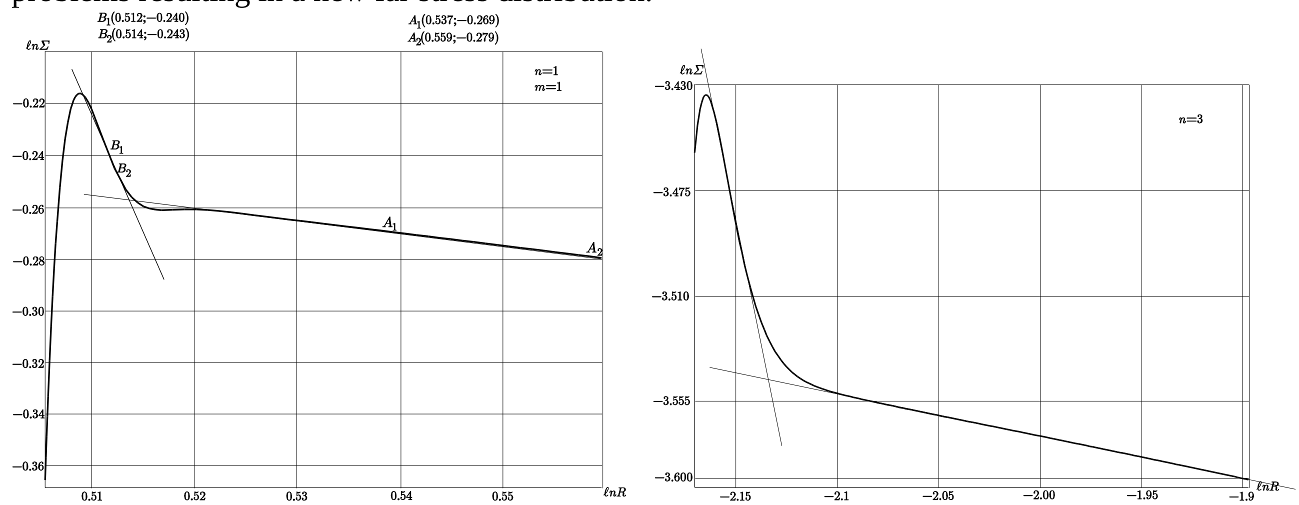

Fig. 3. Logarithmic plot of the effective stress showing that Σ is proportional to the similarity variable R

The configurations of the totally damaged zone governed by the new far stress field are found and analyzed. The new far field stress asymptotics can be interpreted as the intermediate asymptotic valid for times and distances at which effects of initial and boundary conditions on the stress and damage distributions are lost. Higher order fields for damaged nonlinear antiplane shear and tensile crack problems are analytically derived. The higher order fields obtained permit the shape of the totally damaged zone modelled in the vicinity of the crack tip to be determined more exactly. The similarity solutions obtained can be further used in more general multiscaling models in crack tip mechanics which develops multiscale methodologies of crack tip description [15].

References

[1] Z.H. Jin and R.C. Batra: Archive of Applied Mechanics Vol. 68 (1998), p. 247.

[2] H.D. Bui and A. Ehrlacher: Propagation of Damage in Elastic and Plastic Solids. In: Francois D. (Ed.). Advances in Fracture Research, Pergamon Press, Oxford (1980), p. 533.

[3] H.D. Bui: Fracture mechanics: Inverse problems and solutions, Berlin: Springer (2006).

[4] V.I. Astafjev and T.V. Grigorova: Mechanics of Solids Vol. 30 (1995), p. 144.

[5] S. Murakami, T. Hirano and Y. Liu: International Journal of Solids and Structures Vol. 37 (2000), p. 6203.

[6] S. Murakami, Y. Liu and M. Mizuno: Computational Methods Applied Mechanics Engineering Vol. 183 (2000), p. 15.

[7] J. Zhao and X. Zhang: Engineering Fracture Mechanics Vol. 50 (1995), p.131.

[8] H. Riedel: Fracture at High Temperature, Springer, Berlin (1987).

[9] M. Anheuser and D. Gross: Archive of Applied Mechanics Vol. 64 (1994), p. 509.

[10] L.V. Stepanova: Journal of Applied Mechanics and Technical Physics Vol. 49 (2008), p. 142.

[11] G.I. Barenblatt: Similarity, Self-similarity, and Intermediate Asymptotics, Consultants Bureau, New York (1979).

[12] G.I. Barenblatt: Scaling phenomena in fluid mechanics, Cambridge University Press (1994).

[13] P.L. Sachdev: Self-similarity and beyond. Exact solutions of nonlinear problems, New York: Chapman and Hall/CRC (2000).

[14] M. Ciavarella, M. Paggi and A. Carpinteri: Journal of the Mechanics and Physics of Solids Vol. 56 (2008), p. 3416.

[15] G.C. Sih: Theoretical and Applied Fracture Mechanics Vol. 51 (2009), p. 11.