Thermal Fracture of a FGM/Homogeneous Bimaterial with Defects

Vera Petrova[1, a], Siegfried Schmauder[2,b]

[1]Voronezh State University, University Sq.1, Voronezh 394006, Russia

[2]Stuttgart University, Pfaffenwaldring 32, D-70569 Stuttgart, Germany

[a]Vera.Petrova@math.vsu.ru, [b]Siegfried.Schmauder@mpa.uni-stuttgart.de

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Abstract. The main interest of the present work is related to the influence of material inhomogeneities, such as cracks and material non-homogeneous constants, on the stress-strain state in the vicinity of the interface in a bimaterial consisting of a functionally graded material (FGM) and a homogeneous material subjected to a heat flux applied at infinity. It is assumed that thermo-mechanical properties of the FGM are continuous functions of the thickness coordinate. The FGM thermoelastic constants are supposed to have exponential form and Poisson's ratio is assumed to be constant. The uncoupled, quasi-static thermoelastic theory is applicable to this problem so that the solution consists of the determination of the temperature distribution, and the determination of the thermal stresses. The method of the solution is based on the superposition technique and Fourier transform method and leads to singular integral equations. Some fundamental solutions of the problem for some special cases are obtained. Numerical solutions of the singular integral equations can performed for more general cases by the mechanical quadrature method and can be performed to typical material combinations.

Introduction

FGMs are often used as thermal barrier coatings in different engineering structures and are tailored so that to decrease bimaterial mismatch and residual stresses at the interface and prevent delamination, debonding along the interface. Meanwhile, experimental observations show that cracks and defects usually initiate and grow near interfaces (see a review by Noda [1]). These defects are the cause of additional residual stresses near the interface. In this connection in the present work it is studied the thermal fracture of a FGM/homogeneous bimaterial with internal and interface cracks. It is assumed that thermo-mechanical properties of FGMs are continuous functions of the thickness coordinate. To make the problem mathematically tractable, the FGM constants are supposed to have exponential form. In [2] it was shown that the effect of the Poisson's ratio on the SIFs is negligible. Thus it will be assumed that Poisson's ratio is constant.

Different aspects of deformation and fracture investigations of FGMs can be found in the literature, theoretical semi-analytical studies [1-3] and numerical simulations [4]. A general solution of a single and multiple arbitrarily orientated cracks embedded in a non-homogeneous infinite plate under mechanical loading was obtained in paper by Shbeeb et al. [3]. It was assumed that the FGMs have a constant Poisson's ratio and the shear modulus is of an exponential form. The work by Wang et al. [5] is devoted to multiple cracks problems in the FGMs with arbitrarily varying material properties. The algorithm was applied to steady state or transient thermoelastic fracture problems. A laminated composite plate model was used to simulate the material non-homogeneity. In the paper by Guo and Noda [6] a new model, piecewise- exponential model, is proposed to realize the fracture mechanics investigations of the FGMs with arbitrary properties. In this model the FGM is divided into some non-homogeneous layers along the gradient direction of the properties. By using this model the fracture problem of a functionally graded strip with arbitrarily distributed properties and a crack vertical to the free surfaces is studied. The review of the crack problems in FGMs also can be found in this paper [6]. In spite of a lot of literature on the fracture of FGMs many problems of crack interactions remain unsolved, especially in the case of thermal loading.

The main interest of the present work is on the influence of material inhomogeneities, such as cracks and material non-homogeneous constants, on the stress-strain state in the vicinity of the interface in FGM/homogeneous bimaterial subjected by a heat flux.

Statement of the Problem

Let us consider a bimaterial composed of a functionally graded material FGM (denoted by number 1) located at the upper half plane and an homogeneous material (denoted by number 2) located at the lower half plane. The bimaterial is perfectly bonded with the exception of an interface crack of length $2a_0$. The FGM contains N cracks of length $2a_k$. The bimaterial is subjected to a remote heat flux of intensity q (Fig. 1). That is no mechanical forces are applied, only thermal heat flux. The cracks are supposed to be thermo-isolated.

The coordinate system (x, y) is introduced with the x-axis lied along the interface line. The local coordinate system (x_k, y_k) is attached to the internal crack. The crack position is determined by the defect midpoint coordinate (x_k^0, y_k^0) and an inclination angle θ_k of the crack to the interface, i.e. to the x-axis (Fig. 1). It is supposed that all properties of FG material depend only on coordinate y.

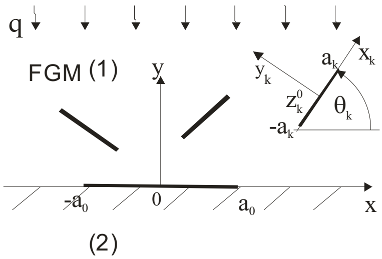

Fig. 1. The geometry of the problem.

The uncoupled, quasi-static thermoelastic theory is applicable to this problem so that the solution consists of the determination of the temperature distribution and the determination of the thermal stresses.

According to the principal superposition, the problem is equal to the superposition of the following two sub-problems: (a) The bimaterial without crack is subjected to remote heat flux, and the heart flux induced at the location x=0 is $q(y)$. (b) The bimaterial with crack is free of remote fluxes and only the crack faces are subjected by heart fluxes of intensity $-q(y)$. The analogous superposition scheme is applied to the thermoelastic problem. The problem (a) is a homogeneous problem of thermo-conductivity and it is not contributed to the singular fields at the crack tips. Problem (b) is called perturbation problem and it governs the singular crack-tip fields. In fracture mechanics the behavior of the singular crack-tip fields are important and considered firstly. In the following only perturbation problem will be analyzed.

The material properties of the FGM can be expressed as follows. The thermal conductivity coefficient is

$$k_1(y) = k_0 e^{\delta y}, \tag{1}$$

where constant k_0 is the thermal conductivity of the interface and for material (2) and δ is the non-homogeneity parameter for the FGM. The Young's modulus and the thermal expansion coefficient can be expressed as

$$E_1(y) = E_0 e^{\beta y}, \quad \alpha_1 = \alpha_0 e^{\gamma y}. \tag{2}$$

Here E_0, α_0 are corresponding constants of homogeneous material (2) and β, γ are non-homogeneity parameters for FGM. In [2] it was shown that Poisson's ratio has not much effect on the solution so that we will assume further ν_j = const (j=1,2).

The relation between global coordinates (x,y) and local coordinate systems (x_k,y_k) can be written in complex form as follows

$$z = z_k^0 + z_k e^{i\theta_k},$$

where $z = x + iy$ and $z_k = x_k + iy_k$. $z_k^0 = x_k^0 + iy_k^0$ is the origin coordinate of the system (x_k,y_k) in the global system. At the same time it is the coordinate of the midpoint of the crack.

In the local coordinate system connected with each arbitrary oriented crack the constant k_1 has the form

$$k_1(x_k, y_k) = k_0 e^{\delta y_k^0} e^{\delta_1 x_k + \delta_2 y_k},$$

where

$$\delta_1 = \delta \sin\theta_k, \quad \delta_2 = \delta \cos\theta_k.$$

The same expressions are written for the other constants.

Thermal problem. Let the temperature field in the bimaterial with cracks be denoted as T_j^* (j=1,2). Due to the principal of superposition it can be presented as

$$T_j^*(x,y) = T_j^0(x,y) + T_j(x,y) \quad (j = 1,2), \tag{3}$$

where $T_j^0(x,y)$ – the temperature distribution in a bimaterial in the absence of cracks, $T_j(x,y)$ – the temperature perturbation caused by the cracks.

In the case of a uniform heat flux applied to the bimaterial the temperature $T_j^0(x,y)$ in an undamaged bimaterial does not cause stresses so that we are interested first of all in the determination of the temperature perturbation $T_j(x,y)$. It should be noted that if a heat source applied at the undamaged bimaterial additional stresses appear and should be calculated.

The heart conduction equation for the steady state is

$$\frac{\partial}{\partial x}\left(k_1(y)\frac{\partial T_1}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_1(y)\frac{\partial T_1}{\partial y}\right) = 0 \tag{4}$$

and taking into account Eq. 1 it is reduced to

$$\Delta T_1 + \delta \frac{\partial T_1}{\partial y} = 0 \tag{5}$$

and for material 2 with $\delta = 0$ we have Laplace equation

$$\Delta T_2 = 0\,, \tag{6}$$

where $\Delta = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$.

The thermal boundary conditions and continuity conditions for the temperature perturbation $T_j(x, y)$ read as follows:

$$k_1 \frac{\partial T_1(x,+0)}{\partial y} = k_2 \frac{\partial T_2(x,-0)}{\partial y} = q_0(x) \quad |\, x \,|\le a_0\,, \tag{7}$$

$$k_j \frac{\partial T_{jn}(x,\pm 0)}{\partial y_n} = q_n(x_n) \quad |\, x_n \,|\le a_n\,, \tag{8}$$

$$k_1 \frac{\partial T_1(x,+0)}{\partial y} = k_2 \frac{\partial T_2(x,-0)}{\partial y}, \quad T_1(x,+0) = T_2(x,-0) \quad |\, x \,|\ge a_0\,, y = 0\,, \tag{9}$$

$$T_1(\pm a_0,+0) = T_2(\pm a_0,-0),\ T_{jn}(\pm a_0,+0) = T_{jn}(\pm a_0,-0) \tag{10}$$

and the temperature perturbation vanishes at infinity so that

$$k_1 \frac{\partial T_1(x, y)}{\partial y} = k_2 \frac{\partial T_2(x, y)}{\partial y} = 0, \quad x^2 + y^2 \to \infty\,, \tag{11}$$

where

$$q_0 = -\frac{\partial T_j^0}{\partial y}, \qquad q_n = -k_j \left.\frac{\partial T_j^0}{\partial y_n}\right|_{y_n=0}.$$

The signs '+' and '–' denote the limiting values of the functions on the upper and lower surfaces of the crack or the interface, respectively.

Thermoelastic problem. The basic equations of plane thermal stress problems for the FGM are the following:
the equilibrium equations

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0, \qquad \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0 \tag{12}$$

and the compatibility condition

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = 2\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y}. \tag{13}$$

Introducing the Airy function U relating to stresses by

$$\sigma_{xx} = \frac{\partial^2 U}{\partial y^2},\ \ \sigma_{yy} = \frac{\partial^2 U}{\partial x^2},\ \ \sigma_{xy} = -\frac{\partial^2 U}{\partial x \partial y}, \tag{14}$$

the governing equation for thermo-elasticity for the FGM part is obtained

$$\Delta\Delta U_1 - 2\beta \frac{\partial}{\partial y}(\Delta U_1) + \beta^2 \frac{\partial^2 U}{\partial y^2} + E_0 \alpha_0 e^{(\beta+\gamma)y}\left(\Delta T_1 + 2\gamma \frac{\partial T_1}{\partial y} + \gamma^2 T_1\right) = 0 . \tag{15}$$

For a homogeneous material $\beta = 0, \gamma = 0$ and we have

$$\Delta\Delta U_2 + E_0 \alpha_0 \Delta T_2 = 0 . \tag{16}$$

The mechanical boundary conditions for the traction-free interface crack are

$$(\sigma_{1y} - i\tau_{1xy})^+ = (\sigma_{2y} - i\tau_{2xy})^- = 0, \ |x| \le a_0, y = 0 \tag{17}$$

and for internal cracks

$$(\sigma_{jy} - i\tau_{jxy})^+ = (\sigma_{jy} - i\tau_{jxy})^- = 0, \ |x_k| \le a_k, y_k = 0 . \tag{18}$$

The continuity conditions in the interface read as follows:

$$(\sigma_{1y} - i\tau_{1xy})^+ = (\sigma_{2y} - i\tau_{2xy})^-, \ |x| > a_0, y = 0, \tag{19}$$

$$(u_1 - iv_1)^+ = (u_2 - iv_2)^-, \ |x| > a_0, y = 0$$

and the condition at infinity is

$$\sigma_{ij} \to 0, \quad x^2 + y^2 \to \infty . \tag{20}$$

The signs '+' and '–' denote as previously the limiting values of the functions on the upper and lower surfaces of the crack or the interface, respectively.

Solution of the Problem

The Fourier transform method is used for the solution of thermal and thermoelastic problems. Due to the superposition principle the problem is decomposed into sub-problems with simple geometry. If we consider a particular case of the interaction of an internal crack normal to the interface ($\theta_k = \pi/2$) with an interface crack ($\theta_k = 0$) we should solve two sub-problems with only one crack and superimpose them.

Consider one of the problems, the thermal problem with an internal crack of length $2a_k$=$2a$ located on the *y*-axis. By applying the Fourier transform to Eq. 5 with respect to y, the following ordinary differential equation is obtained

$$\frac{d^2 f_1(x,\xi)}{dx^2} - (\xi + i\delta)\xi f_1(x,\xi) = 0 \tag{21}$$

where

$$f_1(x,\xi) = \int_{-\infty}^{\infty} T_1(x,y) e^{i\xi y} dy \,. \tag{22}$$

Then, by applying the Fourier transform to Eq. 5 with respect to x, we have

$$\frac{d^2 g_1(\xi,y)}{dy^2} + \delta \frac{dg_1(\xi,y)}{dy} - \xi^2 g_1(\xi,y) = 0 \tag{23}$$

where

$$g_1(\xi,y) = \int_{-\infty}^{\infty} T_1(x,y) e^{i\xi x} dx \tag{24}$$

From Eqs. 22, 23 the solutions of $f_1(x,\xi)$ and $g_1(\xi,y)$ is obtained as

$$f_1(x,\xi) = A_1 e^{-x\sqrt{\xi^2 + i\delta\xi}} + B_1 e^{x\sqrt{\xi^2 + i\delta\xi}} \tag{25}$$

$$g_1(\xi,y) = C_1 e^{(\gamma-\delta)y} + D_1 e^{-\gamma y}$$

where $\gamma = \left(\delta + \sqrt{\delta^2 + 4\xi^2}\right)/2$.

Taking into account the condition at infinity Eq. 11, we get B_1=0.
According to the principal of superposition, the temperature of FGM can be expressed as

$$T_1(x,y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} A_1 e^{-(x\sqrt{\xi^2 + i\delta\xi} + i\xi y)} d\xi + \frac{1}{2\pi} \int_{-\infty}^{\infty} (C_1 e^{(\gamma-\delta)y} + D_1 e^{-\gamma y}) e^{i\xi x} d\xi \tag{26}$$

The following function is introduced as

$$g(y) = \frac{\partial T_1(0,y)}{\partial y} \tag{27}$$

The single-valuedness condition is obtained

$$\int_{-a}^{a} g(t) dt = 0 \tag{28}$$

Applying Fourier transform to Eq. 27 and taking into account condition Eq. 28 we can define that

$$A_1 = \frac{i}{\xi} \int_{-a}^{a} g(y) e^{i\xi y} dy \,. \tag{29}$$

Then applying the Fourier transform to Eq. 26 with respect to *x* and using Eq. 29 we get

$$\overline{T}_1(\xi,y)=\int_{-a}^{a} g(t)\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{i\xi e^{i\lambda(t-y)}}{\lambda(\xi^2+\lambda^2+i\delta\lambda)}d\lambda dt+C_1e^{(\gamma-\delta)y}+D_1e^{-\gamma y} \tag{30}$$

where $\overline{T}_1(\xi,y)=\int_{-\infty}^{\infty} T_1(x,y)e^{i\xi x}dx$.

Now we need a solution of Eq. 6 in the form which is convenient for the construction of the full solution of the problem. Applying the Fourier transform to Eq. 6 with respect to x the following ordinary differential equation is obtained

$$\frac{d^2\overline{T}_2}{dy^2}-\xi^2\overline{T}_2=0 \tag{31}$$

where $\overline{T}_2(\xi,y)=\int_{-\infty}^{\infty} T_2(x,y)e^{\xi x}dx$

The solution of Eq. 31 is

$$\overline{T}_2(\xi,y)=A_2e^{\xi y}+B_2e^{-\xi y} \tag{32}$$

Because of the condition at infinity we have A_2=0. The other unknown constants are obtained from continuity and boundary conditions Eqs. 7-10.

The boundary condition on the crack line can be written as

$$\frac{\partial T_1(0,y)}{\partial x}=q(y)/k_1(y),\quad |y|<a \tag{33}$$

Taking into account Eqs. 26, 29, 33 the following singular integral equation is obtained

$$\frac{1}{\pi}\int_{-a}^{a} g(t)[\frac{1}{t-y}+R(t,y)]dy=q(y)/k_1(y),\quad |y|<a \tag{34}$$

where $R(t,y)$ is the regular kernel, containing the parameters of the problem.

Eq. 34 can be solved numerically by the mechanical quadrature method in which the series of Chebyshev polynomials of first kind is used. The integral equation Eq. 34 together with condition Eq. 28 is reduced to a system of linear algebra equations.

If we are solving the crack interaction problem then we should consider the interface crack problem and superimpose two solutions. The scheme of the solution is the same as described above.

Conclusion

The boundary value problem for the FGM/homogeneous bimaterial is formulated for the interaction between the internal crack with the interface and the interface crack under the influence of a heat flux. The FGM thermoelastic constants are supposed to have exponential form. The uncoupled, quasi-static thermoelastic theory is applicable to this problem so that the solution consists of the determination of the temperature distribution and the determination of the thermal stresses. The solution of a particular problem of two cracks is considered and a method for the solution is presented. The method is based on the Fourier transform and the superposition technique and leads to a system of singular integral equations which can be solved numerically.

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