The Behavior of Statically-Indeterminate Structural Members and Frames with Cracks Present

Paul C. Paris[1, a] and Thierry Palin-Luc[1, b]

[1]Arts et Métiers ParisTech, Université Bordeaux 1,
Laboratoire Matériaux Endommagement Fiabilité et Ingénierie des Procédés (LAMEFIP), EA2727,
Esplanade des Arts et Métiers, F-33405 Talence Cedex, France

[a]pcparis30@gmail.com, [b]thierry.palin-luc@ensam.eu

Keywords: J-integral, plastic hinge, tearing stability analysis, crack stability, elastic plastic crack growth, energy dissipation.

Abstract. Crack stability is discussed as affected by their presence in statically-indeterminate beams, frames, rings, etc. loaded into the plastic range. The stability of a crack in a section, which has become plastic, is analyzed with the remainder of the structure elastic and with subsequent additional plastic hinges occurring. The reduction of energy absorption characteristics for large deformations is also discussed. The methods of elastic-plastic tearing instability are incorporated to show that in many cases the fully plastic collapse mechanism must occur for complete failure

Introduction

Many safety critical structural members and frames incorporate materials with very high resistance to unstable fracture from cracks existing or produced by mechanisms of fatigue, corrosion, etc. Under such circumstances, if the cracked section is later exposed to high loads, it can become fully plastic without crack instability. Nuclear reactor piping, civil buildings and bridges with welded steel beams and frames, offshore oil platforms, etc. are but a few examples. This paper shall assume that it is important to have the cracked section become fully plastic without unstable rapid fracture (the material is tough enough to avoid sudden fracture). It is the objective here to show that for statically-indeterminate members that the conditions for such behavior is often present and can be assured. The approach taken here will be the use of J-Integral based Tearing Stability analysis to demonstrate the case for such behavior.

The J-Integral Tearing Stability Method

The J-Integral is defined in the usual manner with alternate forms [1] by:

$$J = \int_{\Gamma} (W dy - T_i \frac{\partial u_i}{\partial x} ds) = -\int_0^{\delta} \frac{\partial P}{\partial a} d\delta = \int_0^{P} \frac{\partial \delta}{\partial a} dP \tag{1}$$

where: $W = \int_0^{\varepsilon_{ij}} \sigma_{ij} d\varepsilon_{ij}$ and: T_i and u_i are corresponding components of traction and displacement on the contour, Γ, around the crack tip; a is the crack length, δ is the load point displacement and P the load. When the cracked section has reached its limit load and additional plastic deformation the J value at its tip can be approximated very well by:

$$J = -\int_0^{\delta} \frac{\partial P}{\partial a} d\delta = -\int_0^{\theta} \frac{\partial M}{\partial a} d\theta \cong -\frac{\partial M_{Limit}}{\partial a}\theta = J_{applied} \text{ (per unit thickness, } t) \tag{2}$$

where: the limit moment in bending is M_L and θ is the angle of bending of the cracked section. This formula may be used to compute the applied value of $J_{applied}$ for a crack.

The materials resistance to the applied J shall be denoted as $J_{Resistance} = J_R$ and is normally shown on a J_R vs Δa, the change in crack length, diagram, which is called a material's resistance curve. It is an *Equilibrium* statement to say that:

$$J_{applied} = J_R \tag{3}$$

A crack becomes *Unstable* only if: $\dfrac{dJ_{appl}}{dA} \geq \dfrac{dJ_R}{dA}$ where $dA = tda$. This *Unstable condition* is better written:

$$T_{appl} = \frac{dJ_{appl}}{dA}\frac{E}{\sigma_0^2} \geq \frac{dJ_R}{dA}\frac{E}{\sigma_0^2} = T_R \tag{4}$$

which is a more natural non-dimensional form for *Instability* determination [2]. Further, when judging instability it is often convenient to make use of a J vs T diagram. The material's curve on such a J-T diagram can easily be bounded by a rectangular hyperbola for valid data (by ASTM specifications), that is: $J_R \times T_R = C$ (a material constant for a given thickness and temperature). When the J_{appl}, T_{appl} point for a given loading or deformation is below and to the left of the material's curve stability is ensured. Figure 1 shows a schematic of a $J_R - T_R$ curve (J_R vs Δa) and the corresponding material curve on a J-T diagram. These observations can be conveniently used to make stability judgments for cracks with remaining ligaments loaded well into the plastic range.

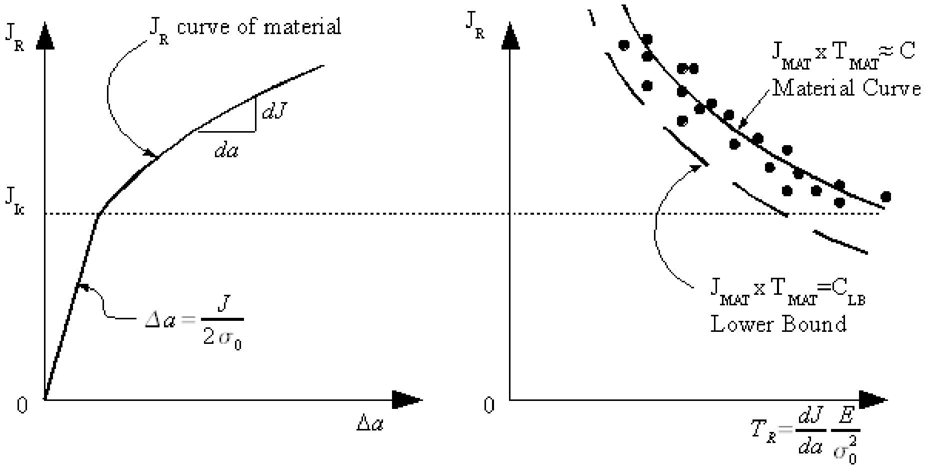

Figure 1: schematic illustration of a J-R curve and a J-T stability diagram.

Stability of a Crack in a Fixed Ended Beam

Figure 2 shows a fixed ended beam with a crack at its center section which has become fully plastic in bending.

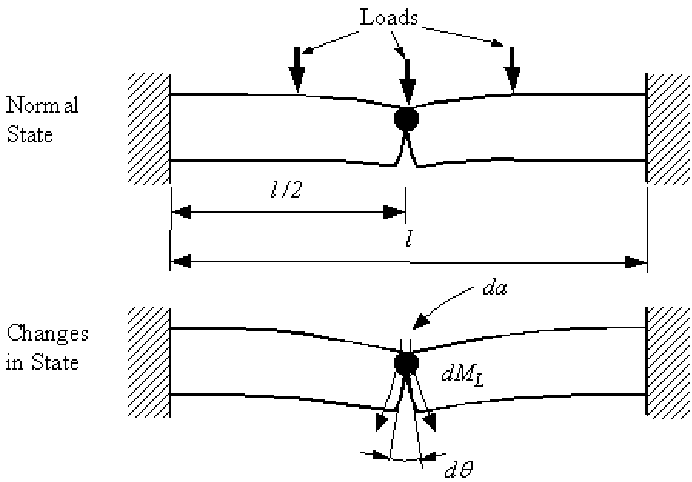

Figure 2: the loaded beam with a plastic hinge at the crack.
(does da cause dM_L ,which causes $d\theta$ and further da)

The peak load is presumed to be applied and the question to be asked is will the crack be stable? If the crack then grows by an increment one should ask whether the reduction in the moment causes an incremental increase in the bend angle at the cracked section, which in turn causes divergent increments of additional crack growth. If only the cracked section is plastic then the additional rotation at the cracked section is provided by further elastic rotation at the cracked section caused only by the change in the limit moment, dM_L. For judging this, simple elastic beam theory gives:

$$d\theta = \frac{dM_L l}{EI} \tag{5}$$

Figure 3 shows some typical cracked cross section as examples. Each one will be analyzed separately to note the pattern of results.

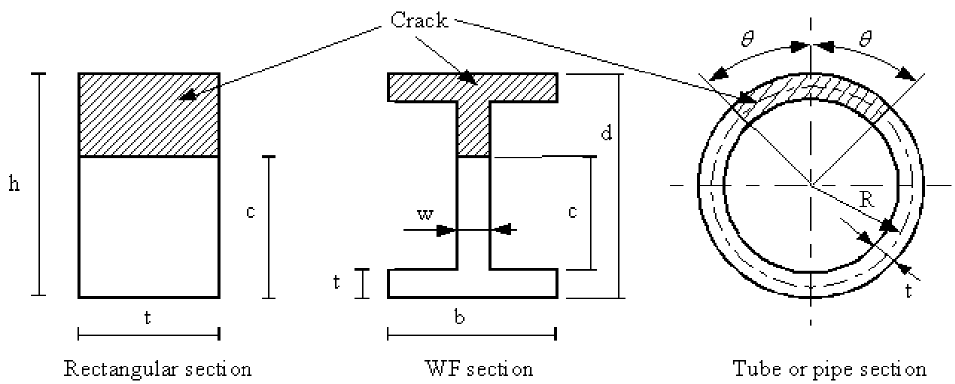

Figure 3: some typical cracked cross sections subjected to fully plastic bending.

The Rectangular Cross-section

For a rectangular cross-section the plane strain solution for the limit moment is the familiar Green and Hundy formula:

$$M_L = 0.35\sigma_0 c^2 t \qquad \text{where} \qquad I = \frac{h^3 t}{12} \tag{6}$$

and σ_0 is the flow stress; h is the section height; t is its width; and c is the remaining uncracked ligament. For this case the applied tearing stability T is:

$$T_{appl} = \frac{E}{\sigma_0^2}\frac{dJ}{tda}\bigg|_{appl} = \frac{E}{t\sigma_0^2}\left[\frac{dJ}{d\theta}\frac{d\theta}{dM}\frac{dM}{da}\right]_{appl} = \frac{E}{t\sigma_0^2}\left[\frac{dM_L}{da}\frac{d\theta}{dM_L}\frac{dM_L}{da}\right] \tag{7}$$

The determination of the $\frac{dJ}{d\theta}$ in the first bracket is from Eq. (2). It is now noted that $da = -dc$ so that then:

$$\left(\frac{dM_L}{da}\right)^2 = 0.49\sigma_0^2 c^2 t^2 \quad \text{and} \quad \frac{d\theta}{dM_L} = \frac{12L_{effective}}{Eh^3 t} \tag{8}$$

where $L_{effective}$ will later be taken as the effective length of a beam equivalent to a fixed ended beam cracked at the center. Combining these results leads to Eq. (9) (for a rectangular beam of height, *h*).

$$T_{appl} \cong 6\left(\frac{c}{h}\right)^2 \frac{L_{eff}}{h} \cong Order(1)\frac{L_{eff}}{h} \tag{9}$$

For a Wide Flanged Section with a Flange Fully Cracked

With a wide flange WF or I section in bending about its strong axis, the flange width is denoted, *b*, its thickness, *t*, the web thickness is *w*, and the total depth is *d.* If the bottom (tension) flange is completely cracked which extends into the web leaving a depth, *c,* of the web uncracked, the analysis proceeds as in the previous section. The limit moment is:

$$M_L = \sigma_0 cw\left(\frac{c+t}{2}\right), \quad \text{where} \quad da = -dc, \quad \text{and} \quad I \cong \frac{btd^2}{2} \tag{10}$$

The web is neglected in the moment of inertia, *I,* and in M_L the *t* is neglected compared to *c* which will be somewhat compensating errors. Combining these results following the steps of the previous section leads to Eq. 11 (for a WF or I section of depth, d).

$$T_{appl} = 2\left(\frac{w}{t}\right)^2\left(\frac{c}{b}\right)\left(\frac{c}{d}\right)\frac{L_{eff}}{d} \cong Order(1)\frac{L_{eff}}{d} \tag{11}$$

The Tube or Pipe Section with a Through Wall Crack

A tube section is taken with a through wall crack extending over a sector of the section of 2θ in size. See Figure 3 again. If the section is of mean radius, R, and thickness, t, (with $t<<R$) and becomes fully plastic then:

$$\frac{dM_L}{da} = 2\sigma_0 th \quad \text{where} \quad h = R\left(\cos\theta + \sin\frac{\theta}{2}\right) \quad \text{and} \quad I = \pi R^3 t \tag{12}$$

Again, proceeding as before (for a tubular section of diameter, D):

$$T_{appl} = \frac{4}{\pi}\left(\cos\theta + \sin\frac{\theta}{2}\right)^2 \frac{L_{eff}}{D} \leq 1.6\frac{L_{eff}}{D} = Order(1)\frac{L_{eff}}{D} \tag{13}$$

A more extensive analyses of cracked piping sections and their effective lengths is given in [3].

The trend noted hear is clear that the T_{appl} for quite different cross sections depends mainly on the beams effective length divided by its height. It is granted that more detail analyses for each section could be computed. Moreover from this simple analysis it has also been demonstrated as the crack gets deeper into each section that the T_{appl} tends to diminish and if a crack does go unstable it is likely to be self-arresting. These trends are emphasized here to encourage further detailed analyses such as [3] which shows nuclear reactor piping cracks are normally always stable even with loads into the plastic range.

The preceding discussion assumed the crack location was at the center of a fixed ended beam. It shall now be appropriate to consider the fully plastic cracked section is l_1 from one end and l_2 from the other end, or $l = l_1 + l_2$. See Figure 4. Using simple beam theory with a hinge at the crack location, the equivalent or effective length can be determined. The result is:

$$EI\frac{d\theta}{dM_L} = L_{eff} = l + \frac{3}{4}\left[\frac{\left(l_2^2 - l_1^2\right)^2}{l_1^3 + l_2^3}\right] \tag{14}$$

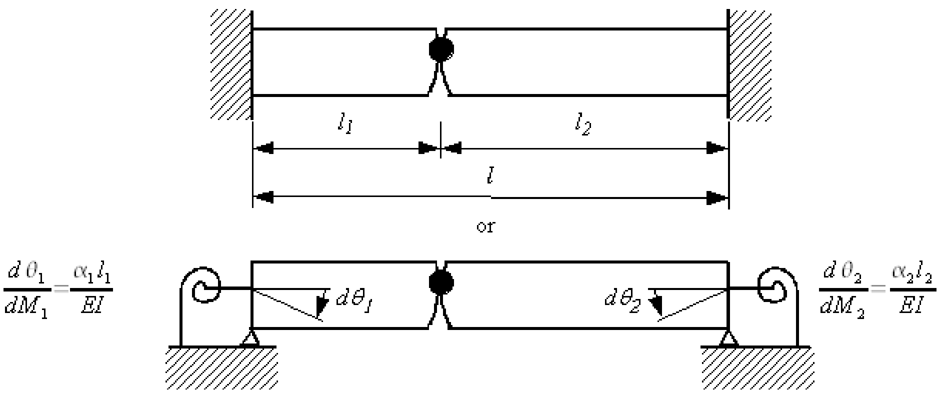

Figure 4: fixed ended or elastically rotationally supported beam ends.

However there is no such thing as a fixed ended beam. Normally a beam is framed into a structure that has elastic stiffness. Let the ends of the beam be framed into joints with rotational compliance equivalent to Eq. 15 where α_1 and α_2 are non-dimensional compliance coefficients.

$$d\theta_1 = \left(\frac{\alpha_1 l_1}{EI}\right) dM_1 \quad \text{and} \quad d\theta_2 = \left(\frac{\alpha_2 l_2}{EI}\right) dM_2 \tag{15}$$

Adding these rotational compliances to the ends of the beam modifies the preceding relationship into the form:

$$EI\frac{d\theta}{dM_L} = L_{eff} = \left[l_1(1+\alpha_1) + l_2(1+\alpha_2)\right] + \frac{3}{4}\frac{\left[l_2^2(1+2\alpha_2) - l_1^2(1+2\alpha_1)\right]^2}{\left[l_1^3(1+3\alpha_1) + l_2^3(1+3\alpha_2)\right]} \tag{16}$$

It is noted that this form reduces to Eq. (14) for $\alpha_1 = \alpha_2 = 0$. It can be used to accommodate various elastic frames supporting the beam.

Further, for $\alpha_1 = \alpha_2 = 0$, let us suppose that an additional plastic hinge forms at the l_2 end of the beam, while the cracked section also remains fully plastic, but with two plastic hinges the collapse mechanism is still not formed. For this case (Figure 5) elastic beam analysis with these hinges gives:

$$EI\frac{d\theta}{dM_L} = L_{eff} = l_1\left(1 + \frac{l_1}{l_2} + \frac{l_1^2}{3l_2^2} + \frac{l_2}{3l_1}\right) \tag{17}$$

It is equally easy to add examples of continuous beams with the second or more plastic hinges short of plastic collapse which is left undone here for more advanced examples.

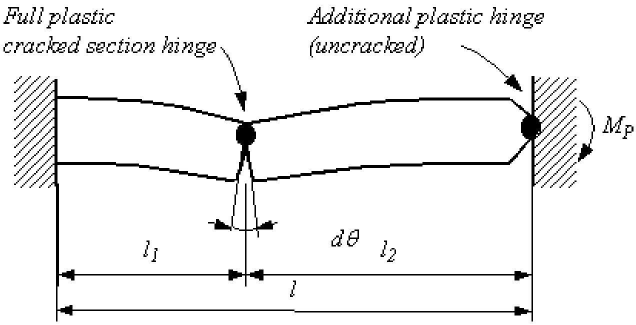

Figure 5: A beam with a plastic hinge in addition to one at the crack.

Some Special Frames and a Ring as Examples of L-effective

Figure 6 shows some frames and a simple ring of interest for further analysis. The two legged frame with differing stiffnesses for the beam, EI_B, and the columns, EI_C, with elastic bending theory leads to Eq. 18 for a centered crack in the beam, l, and a column height, h.

$$EI_B \frac{d\theta}{dM_L} = L_{eff} = l + \frac{EI_B}{2EI_C} h \tag{18}$$

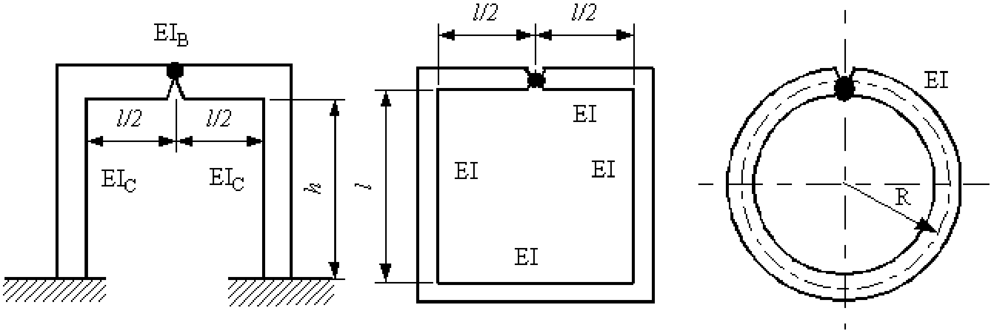

Figure 6: some frames and a ring for $\frac{d\theta}{dM_L}$ determination at the crack.

For the square frame with each side of length, *l,* the result is:

$$EI \frac{d\theta}{dM_L} = L_{eff} = 2l \tag{19}$$

Which is not much of an increase of the effective length over a fixed ended beam. If this was part of a rectangular grid the 2 coefficient would be much closer to 1.

The problem of a ring of radius, *R,* has been discussed in [4] with the crack location at various angles from diametrically opposite concentrated loads. Therein the sequence of formation of plastic hinges was determined and the equivalent of the effective length was also determined for each case. Here the treatment is just with a plastic hinge at the crack location to determine the effective length. It is:

$$EI \frac{d\theta}{dM_L} = L_{eff} = \frac{2\pi}{3} R \tag{20}$$

Again it is noted that the result is a little more than the diameter, *2R,*of the ring.

Finally here the first frame is considered, where typical side-sway induced additional plastic hinges are formed, as shown on Figure 7. Again this problem is analyzed using simple beam theory to obtain:

$$EI_B \frac{d\theta}{dM_L} = L_{eff} = \frac{4}{3} l + \frac{8}{3} \frac{EI_B}{EI_C} h \tag{21}$$

It is also noted here that the stiffnesses of columns in compression with axial loads, *P*, should be adjusted by Eq. 22 where P_{cr} is the Euler critical load in compression.

$$EI_{C-effective} = EI_C\left(1-\frac{P}{P_{cr}}\right) \tag{22}$$

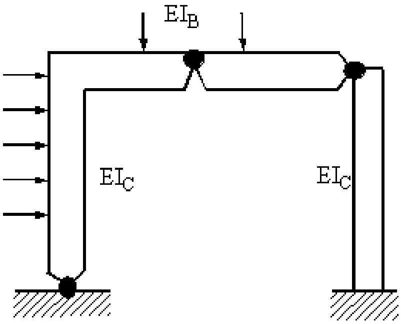

Figure 7: a frame sustaining side-sway loading with two plastic hinges, as well as that at the crack

Material Properties Needed to Assure Crack Stability

The preceding discussion defined the material's tearing modulus as:

$$T_R = T_{material} = \left.\frac{dJ}{dA}\right|_{mat}\frac{E}{\sigma_0^2} \tag{23}$$

where the values which may be used to evaluate crack stability can be determined from a material's *J-R* curve and elastic modulus and flow stress. It was also mentioned that empirically it has been noted that the *J-R* curve valid data will generate a part of a *J vs T* diagram which can be conservatively enclosed by a rectangular hyperbola:

$$J_R \times T_R = J_{mat} \times T_{mat} = C \tag{24}$$

Normally using these items to judge stability for well-designed structures, if cracked, will show very conservatively that crack stability can be achieved. For example consider an infinite sheet with a central crack of $2a$ in length subjected to uniform tension, σ, perpendicular to the crack. This is the Griffith configuration, which if large enough can be treated by the usual elastic analysis. If a J-T diagram for the same thickness of material is bounded by the preceding rectangular hyperbola the following Griffith Equation can be employed:

$$J_{appl} = G = \frac{\pi\sigma^2 a}{E} \tag{25}$$

From this result it follows that:

$$T_{appl} = \frac{dJ_{appl}}{da}\frac{E}{\sigma_0^2} = \frac{\pi\sigma^2}{\sigma_0^2} \tag{26}$$

Combining these two equations leads to:

$$\frac{J_{appl}}{T_{appl}} = \frac{\sigma_0^2 a}{E} \tag{27}$$

which is a straight line of this slope through the origin of the J-T diagram as shown on Figure 8.

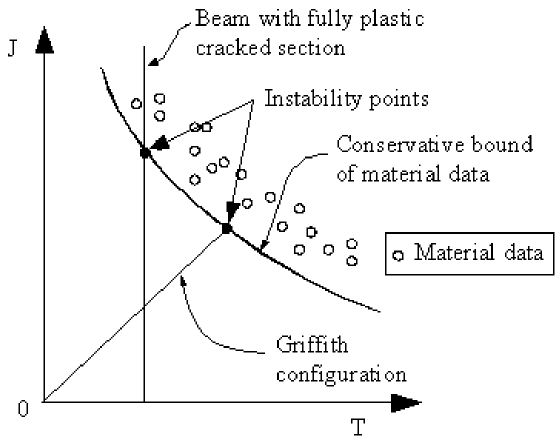

Figure 8: A conservative bound of the material data, with the J_{appl} vs T_{appl} shown for the Griffith configuration and beam analyses.

For the various beam sections treated previously herein the results took the form:

$$T_{appl} = Order(1)\frac{L_{eff}}{D} \tag{28}$$

which are simply a vertical line on the J-T diagram as illustrated on Figure 8. The intersections of these "applied curves" with the "material curve" indicates instability. For the Griffith configuration it is noted that any increase the crack length results in a higher value of J at instability.

For an example in reference [3] it was noticed for typical nuclear piping systems the:

$$\frac{L_{eff}}{D} \cong T_{appl} = 11.2 \text{ to } 62.7 \tag{29}$$

whereas the measured J-Integral tests of this material gave:

$$T_{material} = 205 \text{ to } 452 \tag{30}$$

to very high values of J implying fully plastic action at cracked sections was easily reached with minor crack growth occurring. This practical example gives confidence that these analyses, although approximate are very relevant.

In cases were doubt remains a final method, which does not even employ the concepts of J-Integral Fracture Mechanics can be employed. Experimentally one can take a cracked section of the relevant beam and test it for its moment vs. angle change curve for the cracked section and compare the descending material $\frac{dM_L}{d\theta}$ like those found in the equations above from the structure's L_{eff}.

If the material's $\left.\frac{dM_L}{d\theta}\right|_{material}$ *is less than* the structure's $\left.\frac{dM_L}{d\theta}\right|_{applied}$:

$$\left.\frac{dM_L}{d\theta}\right|_{Mat} < \left.\frac{dM_L}{d\theta}\right|_{appl} \tag{31}$$

then stability is assured. See Figure 9 for an illustration of the test results.

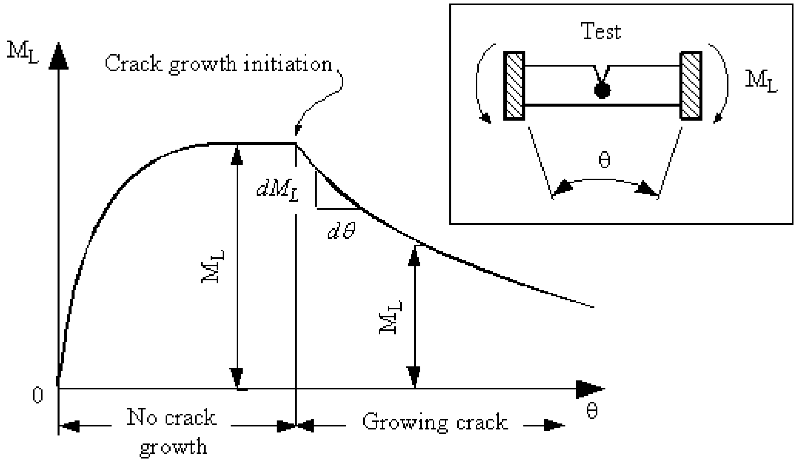

Figure 9: an approach with M_L vs θ testing to avoid a J-Integral approach.

Energy Dissipation by Plastic Collapse Mechanisms with Cracks Present

It is also of interest to compare the work dissipated by plastic hinges including those at cracked sections with those with cracks absent. Figure 10 shows the collapse mechanism for a fixed ended beam (or frame where the collapse mechanism is entirely in one beam).

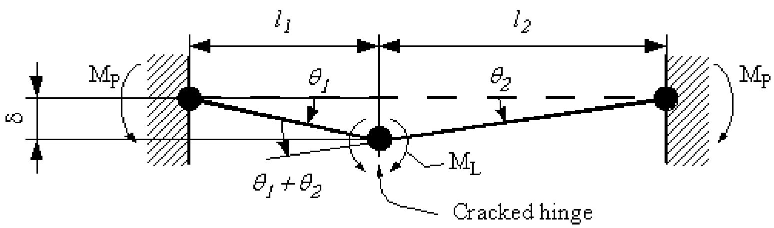

Figure 10: a collapse mechanism for a beam

For the beam in Figure 10 the work done with a crack at the central hinge is:

$$Work(cracked) = (M_P + M_L)(\theta_1 + \theta_2) = (M_P + M_L)\delta\left(\frac{l_1 + l_2}{l_1 l_2}\right) \quad (32)$$

where M_P is the fully plastic hinge moment of the uncracked section, and M_L is the hinge moment of the cracked section. For the same mechanism without a crack at the central location, all the hinge moments are simply M_P. Therefore the ratio of the work cracked to uncracked is:

$$\frac{Work(cracked)}{Work(uncracked)} = \frac{M_P + M_L}{2M_P} = \frac{1}{2}\left(1 + \frac{M_L}{M_P}\right) \quad (33)$$

Now if in Figure 10 the crack location is instead at the l_1 end of the beam then:

$$\frac{Work(cracked)}{Work(uncracked)} = \frac{2l_1 + \left(1 + \dfrac{M_L}{M_P}\right)l_2}{2(l_1 + l_2)} \quad (34)$$

For both of these cases the minimum ratio is 1/2 to a maximum below 1.

Further, consider the previous frame with a side-sway tendency present to develop a final hinge at the base of the other column, as shown in Figure 11, as its collapse mechanism.

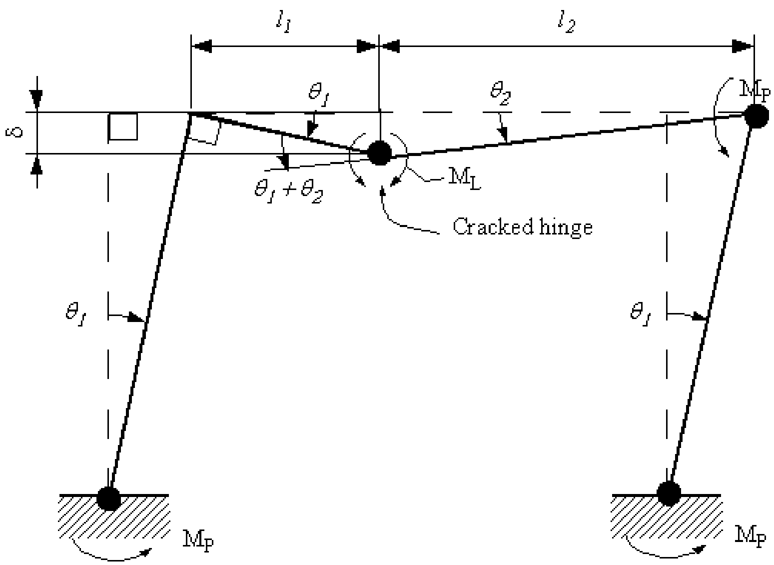

Figure 11: a collapse mechanism for a frame with side-sway.

For this mechanism with the cracked section in the central part of the beam the work is:

$$Work(cracked) = M_P\delta\left[\frac{\left(3+\frac{M_L}{M_P}\right)l_2+\left(1+\frac{M_L}{M_P}\right)l_1}{l_1 l_2}\right] \tag{35}$$

From this setting $M_L = M_P$ the uncracked mechanism can be evaluated to get the work ratio of cracked to uncracked frame as in Figure 11. It is:

$$\frac{Work(cracked)}{Work(uncracked)} = \frac{\left(1+\frac{M_L}{M_P}\right)l_1+\left(3+\frac{M_L}{M_P}\right)l_2}{2l_1+4l_2} \tag{36}$$

The minimum ratio here must be more than 1/2 up closer to 1. Requiring a 4-hinged mechanism has caused that result. However, if not enough side-sway inducing loads are present the beam mechanism of Figure 10 could be relevant instead for this frame.

In order to view these work ratios with a bit more perspective it is relevant to comment on the typical values of $\frac{M_L}{M_P}$ for the various cross sections illustrated on Figure 3 that are listed as follows:

(1) For the *rectangular section* cracked to 1/2 its depth the result is (for plane strain conditions):

$$\frac{M_L}{M_P} = 0.35 \tag{37}$$

(2) For the *wide flange or I section* with the flange fully cracked:

$$\frac{M_L}{M_P} = \frac{1}{2}\left(\frac{w}{t}\right)\left(\frac{c}{b}\right)\left(\frac{c}{d}\right) = 0.28 \quad \text{(for typical proportions)} \tag{38}$$

(3) For a *tube or pipe section* with a through wall crack over a sector $2\theta = \frac{\pi}{2}$ or 90 degrees:

$$\frac{M_L}{M_P} = 0.57 \tag{39}$$

Substituting these values into the previously stated work ratios gives further understanding of the absorption of energy for cracked structures where cracks can remain stable and collapse mechanisms are formed to enhance that absorption.

Conclusions

The application of J-integral tearing stability method to different typical cracked cross sections of beams used in structural mechanical engineering has shown that T_{appl} is approximatively equal to the ratio of an effective length, L_{eff}, over the characteristic dimension of the section (height for a rectangular, diameter for a pipe for examples). Such J-integral analysis together with the simple elastic beam theory applied on some statically indeterminate frames and on a ring with plastic hinges allowed assessment the effective length L_{eff}. Finally, under bending the limit moment for a fully plastic section with a crack, compared to the limit moment for uncracked section, is not negligeable. The absorption of energy for cracked structures can be ensured by collapse mechanisms formed to enhance this absorption. Consequently, with tough material and good design to limit the effective length, as defined herein, it is possible to ensure stable crack growth even if plastic hinges appear. In such a case the fully plastic collapse mechanism must occur for complete catastrophic failure of the structure.

Acknowledgements

The authors acknowledge Arts et Métiers Paris Tech and Foundation Arts et Métiers for the financial support of the Prof. P.C. Paris' stay at LAMEFIP in 2008. The encouragement of Prof. Ivan Iordanoff, Director of LAMEFIP, is also acknowledged with thanks.

References

[1] J. R. Rice, Journal of Applied Mechanics, ASME, Vol.35, June 1968, p.379.

[2] P. C. Paris, et al, Elastic Plastic Fracture (4 papers), ASTM-STP 668, J. D. Landes, J. A. Begley, and G. A. Clarke Eds., ASTM, 1979.

[3] P. C. Paris and H. Tada "The Application of Fracture Proof Design Methods using Tearing Instability Theory to Nuclear Piping Systems Postulating Circumstances of Through Wall Cracks, NUREG/CR-3464,U. S. Nuclear Regulatory Commission, Washington, D. C. , September 1983.

[4] L. J. Young, P. C. Paris, and H. Tada, "The Effects of a Radial Crack and Its Location on the Development of Plastic Hinges and an Analysis of Crack Stability in a Circular Ring with Diametrically Opposite Concentrated Loads" Twenty Second Symposium (Volume I), ASTM, STP1131, H. A. Ernst, A. Saxena and D. L. McDowell, ASTM Philadelphia, 1992, pp.793-808.