Stress State and Limit Equilibrium of Thickness-Inhomogeneous Spherical Shell with a System of Arbitrary Located Surface Cracks

Roman Kushnir, Myron Nykolyshyn, Taras Nykolyshyn, Mykola Rostun
Pidstryhach Institute for Applied Problems of Mechanics and Mathematics,
Ukrainian National Academy of Sciences,
3b Naukova Str., 79060, Lviv, Ukraine, e-mail: kushnir@iapmm.lviv.ua

Keywords: elastico-plastic problem, system of arbitrary located through cracks, singular integral equations, δ_c - model, functionally gradient materials, spherical shell.

Abstract. A thickness-inhomogeneous elastico-plastic spherical shell weakened by a system of arbitrary located through cracks is considered. The shell is under external loading, and self-equilibrated forces and moments may be applied to the crack faces. The cracks do not intersect. On the basis of the δ_c model analogue and distortion method of the theory of shells with cracks the problem is reduced to a system of singular integral equations with unknown limits of integration and discontinuous functions in the right-hand side. An algorithm is proposed for numerical solutions of the system obtained together with plasticity conditions and conditions of stress boundedness near a crack. As an example, a spherical shell made of functionally gradient material is considered, the shell being weakened by four surface cracks. The crack opening vs. loading, geometrical and mechanical parameters is analyzed.

Introduction

The shell constructional elements often operate under conditions when their outer surface is in corrosive surroundings with one physico-chemical parameters, and the inner surface with another. In such cases they are made from layer structures or functionally gradient materials (FGM), i.e. composites with microstructural inhomogeneity and continuously changing mechanical properties along the thickness of a thin-walled constructional element. The need for such materials has been induced, first of all, by cosmic materials science to ensure reliable operation of constructional elements at very high temperatures. The early results of studies on the non-uniform distribution of temperature and stresses caused by it in the bodies of FGM are presented in Ref. [1]. Later on the stressed-strained state, strength, etc., and in particular, stress distribution in cylindrical and spherical shells, hollow cylinder and friction node, caused by thermal or force loading, were studied [2-7]. The influence of physico-mechanical properties of such materials on limit equilibrium of cylindrical [8] and spherical [9] shells weakened by one crack was also investigated.

Statement of the problem

Consider a thickness–inhomogeneous infinite spherical shell weakened by a system of k arbitrary oriented plane-linear surface cracks which do not intersect. As an infinite shell we assume the shell, the boundaries of which are so far that they do not influence the perturbed stressed state caused by the crack. Suppose that the shell is under external load and the forces and moments with equal values but opposite directions can be applied to faces of each surface crack so that during the shell deformation the crack faces do not contact. Assume that the shell material is perfectly elasto-plastic or hardened. We shall confine ourselves to the study of sufficiently deep cracks ($d_m \geq 0.4h$, $2d_m$ is the crack depth ($m = 1, 2, ..., k$), $2h$ is the shell thickness). The material properties, load value and crack sizes are assumed to be such that on their continuation plastic strains develop across the shell thickness as a narrow strip. The material is elastic outside these zones. According to the δ_c - model analogue [10,11] the plastic strain zones are substituted by surfaces of discontinuity of elastic displacements and rotation angles and the response of a plastic strain zone to elastic volume is substituted by the unknown forces and moments which counteract the crack opening. This means that the 3D elasto-plastic problem on limit equilibrium of a spherical shell with a system of k surface cracks of given sizes is reduced to the problem on elastic equilibrium of analogous shell with a system of k fictitious through cracks of unknown length to the faces of which, in addition to the given forces and moments, the unknown ones are applied, satisfying the corresponding plasticity conditions for thin shells [12].

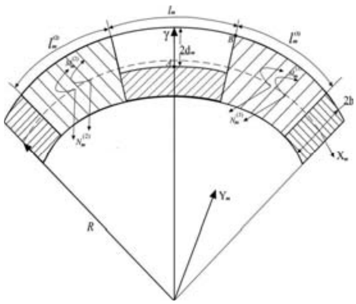

Fig. 1

Refer the median surface of the shell under consideration to the Cartesian coordinate system XOY. Besides, we introduce a local coordinate system $X_m O_m Y_m$ $(m = 1, 2, ..., k)$ on each crack, the origin of which is the center of a fictitious crack and the axis $O_m X_m$ is directed along the crack line. We denote the coordinates of the crack centers in the reference coordinate system XOY by (X_m^0, Y_m^0) and the angles between the axes OX and $O_m X_m$ by β_m. The length of each real surface crack is denoted by l_m and that of a fictitious through crack by l_m^1. In addition, $l_m^1 = l_m + l_m^{(2)} + l_m^{(3)}$, where $l_m^{(2)}$, $l_m^{(3)}$ are the lengths of plastic zones near the left and right tip of the m th crack.

Consider first an infinite spherical shell with one plane-linear fictitious through crack $l_m^{(1)}$. Denote the components of forces and moments arising in the shell with a crack under given loading and under certain boundary conditions by $N_r^1,\ S_r^1,\ M_r^1,\ H_r^1,\ Q_r^1$ with normal and shear forces, bending and twisting moments and cross-resultants, respectively, acting in the normal cross-sections of the shell $r = X_m = \text{const}$ or $r = Y_m = \text{const}$. The components of forces and moments of the basic stressed state caused by the same loading and under the same boundary conditions in the shell without a crack are denoted by $N_r^0,\ S_r^0,\ M_r^0,\ H_r^0,\ Q_r^0$, respectively. Then, taking into account the problem linearity, the forces and moments in the shell with a crack can be given in the form of the sum:

$$G_r^1 = G_r^0 + G_r, \quad G = \{N,\ S,\ M,\ H,\ Q\},$$

where G_r are the components of the perturbed stressed state caused by a crack. These components characterize the stress concentration near a crack.

Since the faces of real crack and, hence, that of fictitious one are also loaded by self-equilibrated forces and moments, then the following conditions for the components of the perturbed stressed state

$$P_i^+(x_m, 0) = P_i^-(x_m, 0) = f_{im}(x_m), \ i = 1, 2, 3, 4 \tag{1}$$

should be satisfied at the crack contour l_m^1. Here

$$f_{im}(X_m)=\begin{cases} P_i^1(X_m)-P_i^0(X_m)+P_i^{(4)}(X_m) & \forall X_m : X_m\in l_m, \\ -P_i^0(X_m)+P_i^{(2)}(X_m) & \forall X_m : X_m\in l_m^{(2)}, \\ -P_i^0(X_m)+P_i^{(3)}(X_m) & \forall X_m : X_m\in l_m^{(3)}; \end{cases}$$

$$P_1=N_2;\ P_2=M_2;\ P_3=S_{21}^*;\ P_4=Q_2^*;$$

the indices "0" and "1" denote the components of the basic stressed state and stressed state of the shell with a crack, respectively; the signs "+" and "–" denote the boundary values of the function on the faces Y_m+0 and Y_m-0; $P^{(2)}$, $P^{(3)}$ are the unknown forces and moments in plastic zones at the crack continuation along the coordinate X_m satisfying the corresponding plasticity conditions of thin shells; $P_i^{(4)}$ are the forces and moments in plastic zone under the crack (Fig.1); S_{21}^*, Q_2^* are the Kirchhoff generalized shear forces and cross-resultants [13].

For the case of loading symmetric about the crack, we assume that in the plastic region under the crack, i.e. in the region $X_m\in l_m$, $\gamma\in[-h;h-2d_m]$ (γ is the coordinate normal to the median surface) constant stresses $\sigma^0=(\sigma_{B1}+\sigma_{T1})/2$ act, where σ_{T1}, σ_{B1} are the integral characteristic of the yield threshold and the strength limit of FGM in the interval $\gamma\in[-h;h-2d_m]$. Then the material response to the break of inner bonds in the plastic zone under the crack is defined as

$$N^l=2(h-d)\sigma^0;\quad M^l=\pm 2d(h-d)\sigma^0;\quad S_{21}^*=Q_2^*=0. \tag{2}$$

Basic relations of thickness-inhomogeneous spherical shell with a crack

The elasticity modulus E and Poisson's ratio μ are the functions of the coordinate γ normal to the median surface

$$E=E(\gamma);\ \mu=\mu(\gamma). \tag{3}$$

The generalized Hooke law and Love hypothesis hold true [13]. The system of key equations for the stress functions $\varphi(x_m,y_m)$ and flexure function $w(x_m,y_m)$, obtained on the basis of the distortions method for thin shells [14], reads:

$$\frac{B}{\left(l_m^1\right)^2}\nabla^2\nabla^2\varphi+\frac{d_{22}}{\left(l_m^1\right)^2}\nabla^2\nabla^2 w-\frac{1}{R}\nabla^2 w=-F_1^0(x_m,y_m);$$

$$\frac{A}{\left(l_m^1\right)^2}\nabla^2\nabla^2 w-\frac{d_{22}}{\left(l_m^1\right)^2}\nabla^2\nabla^2\varphi+\frac{1}{R}\nabla^2\varphi=-AF_2^0(x_m,y_m); \tag{4}$$

where $F_1^0(x_m,y_m)=\nabla^2\left(\varepsilon_{22}^0+d_{22}\kappa_{11}^0+d_{11}\kappa_{22}^0\right)+\nabla^2\left(\varepsilon_{11}^0-\varepsilon_{22}^0+d_{12}(\kappa_{11}^0-\kappa_{22}^0)\right)-\nabla_1\nabla_2\left(\varepsilon_{12}^0+2d_{12}\kappa_{12}^0\right)$;

$$F_2^0(x_m,y_m)=\nabla^2\left(\kappa_{11}^0-\mu\kappa_{22}^0\right)-(1-\mu)[\nabla_2^2\left(\kappa_{11}^0-\kappa_{22}^0\right)-2\nabla_1\nabla_2\kappa_{12}^0];$$

$$d_{11}=\left(C_{11}K_{11}-C_{12}K_{12}\right)/\Omega;\quad d_{22}=\left(C_{11}K_{12}-C_{12}K_{11}\right)/\Omega;\quad d_{12}=K_{66}/C_{66};\quad \Omega=C_{11}^2-C_{12}^2;$$

$$A=D_{11}-D_{11}^0;\qquad B=C_{11}/\Omega;\qquad \mu_1=A_1/A;\qquad A_1=D_{12}-D_{12}^0;\qquad D_{11}^0=K_{11}d_{11}+K_{12}d_{22};$$

$$D_{12}^0 = K_{11}d_{22} + K_{12}d_{11}; \quad C_{ij} = \int_{-h}^{h} B_{ij}(\gamma)d\gamma; \quad K_{ij} = \int_{-h}^{h} B_{ij}(\gamma)\gamma d\gamma; \quad D_{ij} = \int_{-h}^{h} B_{ij}(\gamma)\gamma^2 d\gamma;$$

$$B_{11} = E/1-\mu^2; \qquad B_{12} = \mu B_{11}; \qquad B_{66} = E/(2(1+\mu)); \qquad \nabla^2 = \nabla_1 + \nabla_2; \qquad \nabla_1 = \partial/\partial x_m;$$

$$\nabla_2 = \partial/\partial y_m \; x_m = X_m/l_m^1; \; y_m = Y_m/l_m^1;$$

$[v(x)]$, $[\theta_2(x)]$ are the jumps of displacements and rotation angle; $\delta(y)$ is the Dirac delta function; R is the radius of the shell median surface.

For the shell under consideration with a crack $|x_m| < 1$, $y_m = 0$ the distortion field [14] characterizing the jumps of displacements and rotation angle on the line of m th crack has the following form:

$$\varepsilon_{11}^0(x_m, y_m) = \kappa_{11}^0(x_m, y_m) = 0; \; \varepsilon_{22}^0(x_m, y_m) = \frac{1}{l_m^1}[v(x_m)]\delta(y_m); \; \varepsilon_{12}^0(x_m, y_m) = \frac{1}{l_m^1}[u(x_m)]\delta(y_m);$$

$$\kappa_{22}^0(x_m, y_m) = -\frac{1}{l_m^1}\left\{[\theta_2(x_m)]\delta(y_m) + \frac{1}{l_m^1}[w(x_m)]\delta'(y_m)\right\}; \quad \kappa_{12}^0(x_m, y_m) = -\frac{1}{(l_m^1)^2}\frac{d}{dx_m}[w(x_m)]\delta(y_m); \tag{5}$$

$[\psi(x_m)] = \psi^+(x_m, +0) - \psi^-(x_m, -0) \quad \forall x_m \in l_m^1$, $[\psi(x_m)] = 0 \quad \forall x_m \notin l_m^1$; $\psi = \{u_m, v_m, w_m, \theta_m,\}$; u_m, v_m, w_m are the components of the generalized displacements of the shell median surface.

The components of the stressed state of the shell are defined by the key functions $\varphi(x_m, y_m)$, $w(x_m, y_m)$ using the formulas

$$N_{im} = \frac{1}{(l_m^1)^2}\nabla_j \varphi_m, \; (i \neq j = 1,2); \qquad S = -\frac{1}{(l_m^1)^2}\nabla_1\nabla_2\varphi_m;$$

$$M_{im} = -A\left(\frac{1}{(l_m^1)^2}\nabla_i^2 w_m + \kappa_{ii}\right) - A_1\left(\frac{1}{(l_m^1)^2}\nabla_j^2 w_m + \kappa_{jj}^0\right) + \frac{d_{22}}{(l_m^1)^2}\nabla_i^2\varphi_m + \frac{d_{11}}{(l_m^1)^2}\nabla_j^2\varphi_m;$$

$$H = -A_2\left(\frac{1}{(l_m^1)^2}\nabla_1\nabla_2 w_m + \kappa_{12}^0\right) - \frac{d_{22}}{(l_m^1)^2}\nabla_1\nabla_2\varphi_m;$$

$$Q_i = -\frac{1}{(l_m^1)^3}\nabla_i\nabla^2\left(Aw_m + d_{22}\varphi_m\right) - \frac{1}{l_m^1}\nabla_i\left(A\kappa_{ii}^0 + A_1\kappa_{jj}^0\right) - \frac{A_2}{l_m^1}\nabla_j\kappa_{12}^0; \quad A_2 = 2(D_{66} - K_{66}d_{12}). \tag{6}$$

If we introduce the complex stress function $\Phi(x_m, y_m)$ in the form [15]

$$\Phi(x_m, y_m) = w(x_m, y_m) + i\sqrt{B/A}\,\varphi(x_m, y_m), \tag{7}$$

a system of differential equations (4) is reduced to a key equation:

$$\nabla^2\left(\nabla^2 - \lambda_1^2 s^2\right)\Phi(x_m, y_m) = -R\lambda_1^2 s^2\left(F_1^0(x_m, y_m) - i\sqrt{AB}F_2^0(x_m, y_m)\right), \tag{8}$$

where $\lambda_1^2 = \sqrt{AB} l_1^2 / \left(R(AB + C^2) \right)$; $s^2 = C/\sqrt{AB} + i$; $C = d_{22}$; $i = \sqrt{-1}$.

Using the Fourier integral transform [16] the fundamental solution $\Phi_0(x_m, y_m)$ of equation

$$\nabla^2 \left(\nabla^2 - \lambda_1^2 s^2 \right) \Phi_0(x_m, y_m) = \lambda_1^2 s^2 \delta(x_m) \delta(y_m)$$

is obtained in the form

$$\Phi_0(x_m, y_m) = -\left[K_0(\lambda_1 s\, r_m) - \ln r_m \right] / (2\pi), \tag{9}$$

where $r_m^2 = x_m^2 + y_m^2$; $K_0(z)$ is the Macdonald function.

Integral equations of the problem

On the basis of formula (9), the right-hand side of equation (8) and the convolution operation, we obtain the expressions for $\Phi(x_m, y_m)$ and then, taking into account Eq. (7), the key functions $\varphi(x_m, y_m)$ and $w(x_m, y_m)$

$$q_i(x_m, y_m) = \frac{c_i}{4\pi} \int_{-1}^{1} \left\{ \left[v(\xi) \right] \Phi_i(x_m - \xi, y_m) - \sqrt{AB} \left[\theta_2(\xi) \right] F_i(x_m - \xi, y_m) \right\} d\xi, \ (i = 1, 2), \tag{10}$$

where $q_1(x_m, y_m) = w(x_m, y_m)$; $\quad q_2(x_m, y_m) = \varphi(x_m, y_m)$; $\quad \Phi_1 = f_1 + f_5$, $\Phi_2 = f_2 + f_6$;

$$F_1 = \mu_1 f_6 - (1 - \mu_1) f_2 + (d_{11} f_5 + d_{12} f_1)/\sqrt{AB}, \qquad F_2 = \mu_1 f_5 + (1 - \mu_1) f_1 + (d_{11} f_6 + d_{12} f_2)/\sqrt{AB};$$

$$f_1 = \psi_1 y_m^2/\rho^2 + \psi_3; \quad f_2 = \psi_2 y_m^2/\rho^2 - \psi_4; \quad f_5 = a\Omega_2 - \Omega_1; \quad f_6 = a\Omega_1 - \Omega_2;$$

$$\psi_1 = 4/\left(\lambda_1^2 \rho^2\right) + 2\Omega_4/\left(\lambda_1 \rho\right) - f_5; \ \psi_2 = 2\Omega_3/\left(\lambda_1 \rho\right) - f_6; \ \psi_3 = 2/\left(\lambda_1^2 \rho^2\right) + \Omega_4/\left(\lambda_1 \rho\right);$$

$$\psi_4 = \Omega_3/\left(\lambda_1 \rho\right); \ \Omega_3(\alpha) = \frac{d}{d\alpha}\Omega_1(\alpha); \ \Omega_4(\alpha) = \frac{d}{d\alpha}\Omega_2(\alpha);$$

$$\Omega_1(\alpha) = -i\left[K_0(\lambda_1 s \rho) - K_0(\lambda_1 \bar{s} \rho) \right]; \ \Omega_2(\alpha) = K_0(\lambda_1 s \rho) + K_0(\lambda_1 \bar{s} \rho); \ \rho^2 = z^2 + y_m^2;$$

$$z = x_m - \xi; \quad a = C/\sqrt{AB}; \ \bar{s} = a - i; \quad c_1 = R\lambda_1^2; \quad c_2 = c_1\sqrt{A/B}.$$

Substituting the relations (10) into (6) we obtain the integral representations of forces and moments $N_{1m}, N_{2m}, S_m, M_{1m}, M_{2m}, H_m, Q_{1m}, N_{2m}$ at an arbitrary point of the shell caused by the jumps of displacements and rotation angles along the fictitious crack l_m^1. With these expressions, the forces and moments at an arbitrary cross-section forming some angle $\beta_{nm} (\beta_{nm} = \beta_n - \beta_m)$ with the line of the m th crack are defined by formulas

$$N_{nm} = N_{1m} \sin^2 \beta_{nm} + N_{2m} \cos^2 \beta_{nm} - S_m \sin 2\beta_{nm};$$

$$S_{nm} = \frac{1}{2}(N_{2m} - N_{1m}) \sin 2\beta_{nm} + S_m \cos 2\beta_{nm};$$

$$M_{nm} = M_{1m} \sin^2 \beta_{nm} + M_{2m} \cos^2 \beta_{nm} - H_m \sin 2\beta_{nm};$$

$$H_{nm} = \frac{1}{2}(M_{2m} - M_{1m})\sin 2\beta_{nm} + H_m \cos 2\beta_{nm};$$
$$Q_{nm} = -Q_{1m}\sin\beta_{nm} + Q_{2m}\cos\beta_{nm}. \tag{11}$$

On the basis of formulas (11), (10) and (6), we define the forces and moments on the line of nth crack l_m^1 caused by the jumps of displacements and rotation angles at the crack l_n^1 using the relation between the local coordinate systems

$$x_m = (x_n^0 - x_m^0)\cos\beta_m + (y_n^0 - y_m^0)\sin\beta_m + x_n\cos\beta_{nm} - y_n\sin\beta_{nm};$$

$$y_m = (y_n^0 - y_m^0)\cos\beta_m - (x_n^0 - x_m^0)\sin\beta_m + x_n\sin\beta_{nm} + y_n\cos\beta_{nm}.$$

If we sum the forces and moments on the line of nth crack caused by the jumps of displacements and rotation angles of each of k cracks and demand that those sums satisfy the given conditions (1) at the crack l_m^1, then we obtain a system of four integral equations in $4k$ unknown functions. Having constructed such equations for each of k cracks we obtain a system of $4k$ singular integral equations to define $4k$ unknown functions characterizing the jumps of displacements and rotation angles along these cracks.

This system reads

$$\int_{l_n^1}\frac{\psi_{in}(s)ds}{s - x_n} + \sum_{j=1}^{4}\sum_{m=1}^{n}\int_{l_m^1}\psi_{jm}(s)K_{nm}^j(s, x_m)ds = d_{in}f_{in}^*(x_n),$$
$$x_n \in l_n^1 \quad (n = 1,\dots,k;\ i = 1,2,3,4). \tag{12}$$

Here $f_{in}^* = f_{in}(n = 1,2,3)$; $\quad f_{4n}^* = f_{4n} + C_n$; $\quad \psi_{1n}(s) = \dfrac{d}{ds}[v_n(s)]$; $\psi_{2n}(s) = \dfrac{d}{ds}[u_n(s)]$;

$$\psi_{3n}(s) = -Rc\frac{d}{ds}[\Theta_{2n}(s)] \;\; \psi_{4n}(s) = \frac{d}{ds}\left[\frac{dw_n(s)}{ds}\right];\; c^2 = \frac{h^2}{R^2\sqrt{3(1-\upsilon^2)}};$$

$K_{nm}^j(s, x_n)$ are continuous functions for the whole set of real values s, x_n; d_{in} are the given constant values; C_n are the integration constants.

The solutions to the system (12) should satisfy the conditions:

$$\int_{l_n^1}\psi_{1n}(s)ds = 0 \;\; (i = 1,2,3,4) \;;\; \iint_{l_n^1}\psi_{4n}(s)dsds = 0,\ (n = 1,\dots,k), \tag{13}$$

ensuring the uniqueness of displacements and angles of rotation at the tips of fictitious cracks.

Note that in the system of integral equations (12) the limits of integration are unknown, since the sizes of plastic zones are unknown, and, hence, the lengths of fictitious cracks are unknown too. Besides, the functions $f_{in}^*(x_n)$ contain the unknowns $N_m^{(i)}, S_m^{(i)}, M_m^{(i)}, Q_m^{(i)}$ ($i = 2,3$) acting at the faces of fictitious cracks as response of plastic zones to the shell elastic material. Hence, the system (12) should be supplemented by some additional conditions in order to construct the solution. It would appear natural that the plasticity conditions and conditions of the stress boundedness in plastic zone (near the tips of fictitious cracks) are the conditions which $N_m^{(i)}, S_m^{(i)}, M_m^{(i)}, Q_m^{(i)}$ ($i = 2,3$) should satisfy.

The solution of system (12) is constructed using the generalization of a variant (proposed earlier [17]) of the method of mechanical quadratures for solving one equation. The functions $f_{in}^{*}(x_n)$ contained in the right-hand sides of system (12) have discontinuities at points $s = x_n^*$ (x_n^* are the coordinates of the tip of the real crack). Comparison of analytic solutions of the system of canonical singular equations corresponding to the system (12) ($K_{nm}^{j}(s, x_n) = 0$) with the corresponding solutions obtained using the method of mechanical quadratures has shown that application of this method directly to the systems with discontinuous right-hand sides leads to substantial errors in the solution in the vicinity of points $s = x_n^*$, where the behavior of the solution is of great interest. Therefore, when constructing the solutions to the systems of type (12), the method proposed for one equation of analogous type in [18] is applied. For this purpose the unknown functions ψ_{jm} are given in the form of the sum

$$\psi_{jm}(s) = h_{jm}(s) + F_{jm}(s) . \tag{14}$$

Here $h_{jm}(s)$ is the solution to a canonical system of equations

$$\int_{l_n^1} \frac{h_{in}(s)ds}{s - x_n} = d_i f_{in}^{*}(x_n),\ x_n \in l_n^1 \quad (n = 1,\dots,k;\ i = 1,2,3,4) \tag{15}$$

obtained using the Cauchy-type inversion formulas [11] and satisfying the conditions of Eq. (13) type. Substituting (14) into (12) and accounting for (15), we obtain a system of singular integral equations for $F_{jm}(s)$. The system obtained is of the form (12) but with continuous right-hand sides.

As an example, we consider the limit equilibrium of thickness-inhomogeneous spherical shell weakened by four cracks of the same length $2l_0$ and depth $2d_0$ equidistant from the origin of coordinates xoy. The cracks form the angles $\beta_n = 2\pi / k(n-1)$, $n = \overline{1,4}$, $k = 4$ with the axis ox. Let the shell be under internal pressure of intensity p. The crack faces are load-free ($P_i^1(x_m) = 0$ $(i = \overline{1,4})$). Among the components of the basic stressed state only $N_2^0 = qR/2$ are different from zero. Then all the cracks are under equal conditions. The unknown zones of plastic strains near all the cracks are the same and are equal to $l^{(2)}$ near the closer (to the origin of coordinates) crack tips and to $l^{(3)}$ near the remote ones. Among the forces and moments acting in the plastic zone, $N^{(2)}, N^{(3)}, M^{(2)}, M^{(3)}$ are different from zero. Therefore, from system (12) we obtain a system of two singular integral equations to find the unknown jumps of displacement v and rotation angle θ_2:

$$\sum_{n=1}^{2} \int_{-1}^{1} \Psi_n(s) K_{in}(x,s) ds = f_i^*(x),\ i = 1,2;\ |s| < 1, \tag{16}$$

where $\Psi_1(s) = \frac{d}{ds}[v(s)]$, $\Psi_2(s) = -\frac{d}{ds}[\theta_2(s)]$;

$$K_{ij}^* = \sum_{k=1}^{2}\left[-\frac{d_{ij}}{z_k} + K_{ij}^0(z_k)\right] + 2K_{ij}^0(s,t);\ K_{ii}^0(s,t) = \varphi_i(s,t);$$

$$K_{13}^0(s,t) = (1-\upsilon)\varphi_2(s,t) - \lambda \kappa ei'(\lambda\rho)\frac{s}{\rho} - \lambda^2 \int_{1}^{-s} \kappa er'(\lambda\rho) d\rho;$$

$$K_{31}^0(s,t) = (1-\upsilon)\varphi_2(s,t) - \lambda \kappa ei'(\lambda\rho)\frac{s}{\rho};$$

$$K_{33}^{0}(s,t)=-(1-\upsilon)^2\varphi_1(s,t)-\upsilon\lambda^2\int_t^{-s}\kappa ei'(\lambda\rho)d\rho;\ \rho^2=s^2+t^2;\ s=-\left(\xi+\frac{d}{l_1}\right);$$
$$t=\frac{x}{l}+\frac{d}{l};\ z_1=x-\xi;\ z_2=-\left(x+\xi+2\frac{d}{l}\right);\ d_{11}=-1;\ d_{12}=d_{21}=-a_1;\ d_{22}=a_2;$$
$$\varphi_1=\left(\frac{8\kappa ei'(\lambda\rho)}{\lambda\rho}-4\kappa er'(\lambda\rho)+\lambda\rho\kappa er'(\lambda\rho)\right)\frac{st^2}{\rho^4}-\left(\frac{\kappa ei'(\lambda\rho)}{\lambda\rho}-\kappa er'(\lambda\rho)\right)\frac{s}{\rho^2};$$
$$\varphi_2=\left(\frac{8}{\lambda^2\rho^2}+\frac{8\kappa er'(\lambda\rho)}{\lambda\rho}+4\kappa ei'(\lambda\rho)-\lambda\rho\kappa ei'(\lambda\rho)\right)\frac{st^2}{\rho^4}-\left(\frac{2}{\lambda^2\rho^2}+\frac{2\kappa er'(\lambda\rho)}{\lambda\rho}+\kappa ei'(\lambda\rho)\right)\frac{s}{\rho^2};$$
$$f_{11}^{*}(x)=-\frac{2\pi}{D_0}f_{11};\ f_{22}^{*}(x)=\frac{2\pi Rc}{D_1}f_{22}(x).$$

d is the distance from the center of fictitious crack to the origin of coordinates.

The solution to the system obtained is sought in the form (14), whereas the solution of the system of integral equations for functions $F_{jm}(s)$ $(j=1,2;m=1)$ is found using the method of mechanical quadratures [17]. This system is solved together with boundedness conditions for normal force and bending moment near the crack tips; for this purpose it suffices that corresponding stress intensity factors at the fictitious crack tips are equal to zero, i.e.

$$K_N(d_2)=K_N(d_3-l^{(2)})=K_M(d_2)=K_M(d_3-l^{(2)})=0. \tag{17}$$

The Tresca plasticity conditions in the form of a plastic hinge

$$\left(\frac{N^{(i)}}{2h\sigma_T}\right)^2+\frac{\left|M^{(i)}\right|}{h^2\sigma_T}=1,\qquad i=2,3, \tag{18}$$

or plasticity condition of a surface layer

$$\frac{N^{(i)}}{2h\sigma_T}+\frac{3\left|M^{(i)}\right|}{2h^2\sigma_T}=1,\qquad i=2,3 \tag{19}$$

are employed.

In relations (17)-(19) the following notations are used: d_2 is the distance from the point of origin to the nearest tip of fictitious crack; $d_2=d-(l^{(2)}+l^{(3)}+2l_0)/2$; $d_3=d_2+l^{(2)}+2l_0$; σ_T is the yield limit of the shell material. In the plasticity conditions (18), (19) it is assumed that $N^{(i)}=const$, $M^{(i)}=const$, i.e. that the shell material is ideally elasto-plastic. If hardening of shell material is taken into account, then we assume that $N^{(i)}$ and $M^{(i)}$ in the plastic zones $l^{(i)}$ $(i=2,3)$ vary according to the linear law. So, for example, in plastic zones in the vicinity of the tips of the right crack ($\beta_n=0$) the forces $N^{(i)}$ and moments $M^{(i)}$ ($i=2,3$) take the form

$$N^{(2)}(x)=P\left[(x-d_2)(m^*-1)/l^{(2)}+1\right];\quad N^{(3)}(x)=P\left[(x-d_3)(1-m^*)/l^{(3)}+m^*\right],$$
$$M^{(2)}(x)=H\left[(x-d_2)(m^*-1)/l^{(2)}+1\right];\quad M^{(3)}(x)=H\left[(x-d_3)(1-m^*)/l^{(3)}+m^*\right], \tag{20}$$

where $m^* = \sigma_B / \sigma_T$; σ_B is the strength limit of the shell material ; P, H are the unknown constants which have to satisfy the given plasticity condition, e.g. the condition of a plastic hinge

$$\left[P/(2h\sigma^{(i)}) \right]^2 + |H|/(h^2\sigma^{(i)}) = 1, \; i = 2,3 . \tag{21}$$

Then

$$\frac{N^{(i)}(x)}{2h\sigma^{(i)}(x)} + \frac{M^{(i)}(x)}{h^2\sigma^{(i)}} = 1, \; i = 2,3 \tag{22}$$

$$\sigma^{(2)}(x) = (x - d_2)(\sigma_B - \sigma_T)/l^{(2)} + \sigma_T ; \quad \sigma^{(3)}(x) = (x - d_3)(\sigma_T - \sigma_B)/l^{(3)} + \sigma_B .$$

The method of mechanical quadratures reduces the solution of the system of singular integral equations to a system of algebraic equations. It should be noted that the unknown sizes of plastic strain zone enter into the system of algebric equations nonlinearly. Therefore, the algorithm of solution is as follows. Having assigned in a certain way $l^{(2)}$ and $l^{(3)}$, we solve the system of linear algebraic equations. Using (17) we find $N^{(i)}$, $M^{(i)}$, $i = 2,3$ and verify the plasticity conditions (18), (19) or (20)-(22). If they are satisfied with the preassigned accuracy, then the problem is solved. In the opposite case we change $l^{(2)}$ and $l^{(3)}$ and the procedure is repeated.

Integrating the obtained solution to the system of integral equations, the crack opening $\delta(x_m, \gamma)$ at its arbitrary point is defined by the formula $\delta(x_m, \gamma) = [v(x_m)] + \gamma[\theta_2(x_m)]$.

Numerical analysis

Numerical analysis is carried out for a spherical shell with such parameters $h/R = 0{,}01$; $\mu = 0{,}3$; $\sigma_B / \sigma_T = 1{,}5$; $d_0 / h = 0{,}5$; $l_0 / h = 10$; $\sigma_{T1} / \sigma_T = 1{,}3$; $N_2^0 = Rp/2$, $M_2^0 = 0$. The shell is made of FGM, the outer surface of which is aluminum ($E_3 = 70$ GPa) and the inner one is germanium ($E_B = 151$ GPa). The elasticity modulus $E(\gamma)$ varies along γ according to the law [6]

$$E(\gamma) = (E_3 - E_B)V + E_B , \; V = \left(\xi + \frac{1}{2} \right)^m , \; \xi = (\gamma/2h); \quad \mu(\gamma) = const . \tag{23}$$

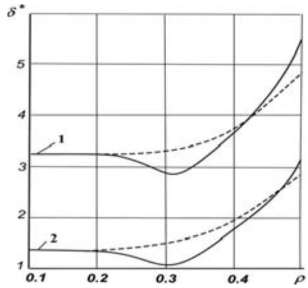

Fig. 2.

Fig. 2 shows the relative crack opening $\delta^* = \delta E / (l_0 \sigma_T)$ at the point B vs. the parameter $\rho = l_0 / d$ (Fig. 1). The curves 1 and 2 correspond to the relative loading $p / \sigma_T = 0.008$ and 0.012. The solid lines correspond to the crack tips located nearer to the origin of coordinates and the dashed lines correspond to the remote tips. The crack opening has also been defined at the point A. Its dependence on ρ has analogous nature, but the value is greater by 20 %.

It follows from the graphs that the beginning of interaction between four cracks located symmetrically in a spherical shell depends both on the crack length (or distance d between them) and on the level of internal pressure p. In particular, for $p = 0.008\sigma_T$ the interaction starts at $\rho = 0.25$ and for $p = 0.012\sigma_T$ at $\rho = 0.2$. At the beginning of interaction, as in the case of

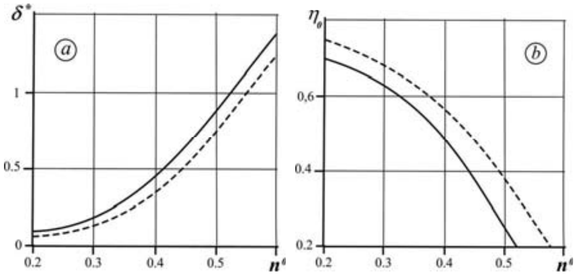

Fig. 3.

homogeneous shell with two through cracks [10], the opening of the crack tips nearer to the origin of coordinates decreases slightly. Next, when cracks approach, it increases rapidly. The results obtained using the plasticity conditions (18) or (19) differ not more than by 5 %. Taking into account hardening of shell material (conditions (20)-(22)) causes the decrease of the crack tips opening. For example, for $m^* = 1,2$ at $\rho = 0,5$ the crack opening has decreased by 10 % and for $m^* = 1,5$, $\rho = 0,5$ by 14 %.

Fig. 3*a* presents the relative crack front opening δ^* at the point A vs. relative loading $n^0 = Rp/(4h\sigma_T)$, and Fig. 3*b* shows the relative length of plastic strips near the crack tip $\eta_0 = l_0/l_1$ vs. n^0. Calculations have been made for $\rho = 0,15$ with other parameters being the same as in Fig. 2. The solid lines correspond to the plasticity condition (18), and the dashed ones correspond to conditions to (20)-(22) for $m^* = 1,5$. Numerical results obtained for conditions (18) differ less by 3 % than those for condition (19).

Conclusion

If critical opening of the crack front is taken as the fracture criterion, then for the considered loading, geometric and mechanical parameters fracture of a spherical shell made of FGM weakened by surface cracks will start at the crack depth continuation extension, i.e. at the point A. The law of distribution of the elasticity modulus $E(\gamma)$ along the shell thickness influences insignificantly its limit equilibrium, in contrast to the ratio σ_3/σ_B. The plasticity condition influences insignificantly the surface crack faces opening and the length of plastic strips near their tips.

References

[1] [1] Koizumi M. The concept of FGM // *Ceramic Transactions, Functionally Gradient Materials*. – 1993. – 34. – P. 3-10.

[2] [2] Y.Obata, N.Noda *Steady thermal stresses in a hollow circular cylinder and a hollow sphere of a functionally gradient material* // J. Thermal Stresses. – 1994. – 17. – P. 471-87.

[3] [3] K.S.Kim, N.Noda *Green's function approach to unsteady thermal stresses in an infinite hollow cylinder of functionally graded material* // Acta Mech. – 2002. – 156. – P. 145-61.

[4] [4] K.S.Kim, N.A.Noda *Green's function approach to the deflection of a FGM plate under transient thermal loading* // Arch. Appl. Mech. – 2002. – 72. – P. 127-37.

[5] [5] Z.S.Shao, L.F.Fan, Wang T.J. *Analytical solutions of stresses in functionally graded circular hollow cylinder with finite length* // Key Eng. Mater. – 2004. – P. 261-263.

[6] [6] M.Ruhi, A.Angoshtari, and R.Naghdabadi *Thermoelastic analysis of thick-walled finite-length cylinders of functionally graded materials* // J. Thermal Stresses. – 2005. – 28. – P 391-408.

[7] [7] R.Kushnir, B.Protsiuk, V.Syniuta *An approach to determining temperature and displacements of inhomogeneous friction couples* // Proc. of the Sixth International Congress on Thermal Stresses. – Vienna University of Technology, 2005. – P. 733-736.

[8] [8] R.M.Kushnir, T.M.Nykolyshyn & M.Yo.Rostun *Limit equilibrium of thickness-ingomogeneous shperical shell with a surface crack* // Mashynoznavstvo, 2006. № 5, – P. 3-7.(in Ukrainian).

[9] [9] R.M.Kushnir, T.M.Nykolyshyn and M.Yo.Rostun *Limit equilibrium of a FGM cylindrical shell with a through crack* // Phizyko-chimichna mekhanika materialiv – 2007. – 43, № 3., – P. 5 – 11.(in Ukrainian).

[10] [10] V.VPanasyuk *Mechanics of quasi-brittle destruction of materials*. Kiev: Naukova Dumka (1991) (in Russian).

[11] [11] R.M.Kushnir, M.M. Nykolyshyn and V.A.Osadchuk *Elastic and elasto-plastic limit state of shells with defects*. L'viv: "SPOLOM" (2003) (in Ukrainian).

[12] [12] V.Prager *Problems of elasticity theory*. Moscow: Phizmatgiz (1958) (in Russian).

[13] [13] S.A. Ambartsumian *General theory of anisotropic shells*. – Moscow: Nauka, 1974. – 446c. (in Russian).

[14] [14] PYa.S.idstrygach, V.A.Osadchuk, E.M.Fediuk and M.M.Nykolyshyn *Method of distortion in the theory of thin shells with cracks. Math. Meth. Phys.-Mech. Fields*. vol. **1**: 29-41 (1975) (in Russian).

[15] [15] V.V.Novozhylov *Theory of thin shells*. – L.: Sudromgiz, 1962. – 431p. (in Russian).

[16] [16] N.Sneddon Fourier transform. – Moscow: Izd-vo inost. lit., 1955. – 668c. (in Russian).

[17] [17] A.I.Kalandija *Mathematical methods of two-dimensional elasticity*. Moscow: Nauka (1973) (in Russian).

[18] [18] N.I.Loakimidis *The numerical solution of crack problems in plane elasticity in the case of loading discontinuities* // Eng. Fract. Mech. – 1980. – 13, No. 4. – P. 709–716.