RMS SIF Weight Functions for Surface Cracks: an Illustration of Capabilities and Limitations

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Nomenclature

a	surface crack depth
c	surface crack half length
C	Paris law coefficient
K	stress intensity factor (SIF)
m	Paris law coefficient
RMS	root mean square
WF	weight function

Abstract. This paper discusses the problem of surface crack growth and its modelling. The concept of root mean square stress intensity factors (RMS SIF) is discussed for the general class of semi-elliptical surface cracks and its advantage over the traditional two point (or multi-point) SIF approach in conjunction with the Paris law is emphasised. A novel technique for the derivation of the RMS SIF weight functions for surface cracks is presented and results are compared with numerical solutions for a variety of loadings and geometries.

1. Introduction

Surface cracks account for the majority of structural fatigue failures. Cracks usually initiate from surface defects, which then develop into a part-through crack. Several observations have shown that these cracks are usually semi-elliptical in shape and that in flat specimens these cracks tend to retain a semi-elliptical shape during their growth [1,2]. Like edge cracks, the stable growth stage of surface cracks accounts for a considerable portion of the propagation life, which fortunately makes their inspection more likely.

Linear Elastic Fracture Mechanics has been successfully applied to quantify growth rates of cracks under cyclic loads. The Paris law [3] can be applied to edge cracks and is also used for surface cracks. Traditionally these cracks are modelled using the Paris law where the size of the crack is determined by two (or more) apparently independent characteristic dimensions. These dimensions can be the depth and half length of the crack, or several characteristic dimensions for which Paris law is applied separately for each point. This has been expressed mathematically in terms of the surface and the deepest points:

$$\frac{da}{dN} = C_{DP}\left(\Delta K_{DP}\right)^m \quad ; \quad \frac{dc}{dN} = C_{SP}\left(\Delta K_{SP}\right)^m . \tag{1}$$

However, as early as the 1970s it was observed that the two material constants C_{DP} and C_{SP} are not equal [4]. This can be attributed to the fact that the stress state in these cracks varies from plane strain at the deepest point to plane stress and the surface and that the plastic zone is larger at the surface points.

Cruse and Besuner [5] were the first to utilise the concept of an integrated average of the stress intensity factor in what is now known as the Root Mean Square (RMS) Stress Intensity Factor (SIF). RMS SIF is defined, for the two principal growth dimensions, as:

$$\overline{K}_x^{\,2} = \frac{1}{\Delta A_x}\iint_{\Delta A_x} K^2(s)dA \text{ and } \overline{K}_y^{\,2} = \frac{1}{\Delta A_y}\iint_{\Delta Ay} K^2(s)dA \qquad (2)$$

where $\Delta A_x = \pi a_y \Delta a_x$ and $\Delta A_y = \pi a_x \Delta a_y$.

Their method involves definition of a number of characteristic dimensions (usually two) for a crack; the crack propagation being described by keeping track of these dimensions. For the crack shown in Fig. 1, these parameters are a_x and a_y which denote crack lengths in the two perpendicular dimensions, as shown. Cruse and Besuner [5] assumed that the coefficients of the Paris law for this type of analysis are the same as for when normal stress intensity factor values (i.e. K) are used.

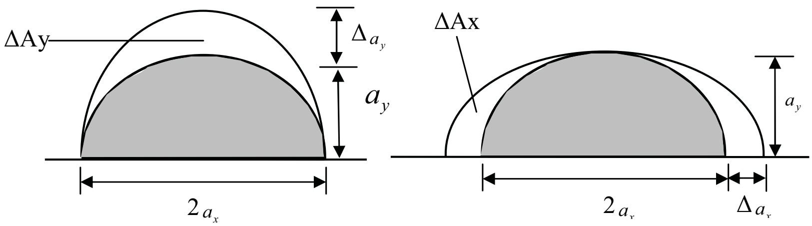


Fig. 1: Two characteristic growth dimensions

Hence the Paris law for surface crack growth can be written as:

$$\frac{da}{dN} = C_A\left(\Delta K_{RMS,A}\right)^m \text{ and } \frac{dc}{dN} = C_B\left(\Delta K_{RMS,B}\right)^m. \qquad (3)$$

Unlike the multi-point approach, it is experimentally observed by various authors that, in many cases $C_A = C_B$. For example see the works of Mahmoud [6,7].

2. Stress Intensity Factor Calculation for Surface Cracks

For the cases of surface cracks appearing on plates under cyclic tensile or bending loads, Newman and Raju [8] have derived an empirical formula for the spatial variation of stress intensity factor based on a large set of three-dimensional finite element analyses. It is, however, sometimes the case that surface cracks should propagate in stress fields that are not uniform or linear; residual stress fields are a good example for this.

First introduced by Beuckner in 1970 [9], the concept of weight functions has since been a well established tool for SIF calculation for edge and through-cracks under arbitrary loads. Beuckner showed that for a two-dimensional crack problem, the stress intensity factor can be expressed as a function of the applied stress as:

$$K = \int_0^a \sigma(x) h(a,x) dx \tag{4}$$

Where 'a' is the crack length and *h* is the weight function. Rice [10] showed that the stress intensity factor can be expressed in terms a reference stress intensity factor and the spatial derivative of the corresponding displacement filed as:

$$h(a,x) = \frac{H}{2K}\frac{\partial \mathbf{u}}{\partial a} \tag{5}$$

Where *H* is an elastic constant. This technique requires numerical differentiation of the crack-face displacement field, which as demonstrated by Petroski and Achenbach [11] and Fett [12], can be troublesome as differentiation of discrete numerical value is both cumbersome and can lead to unstable results.

Therefore, in order to reduce the number of computations needed to obtain the weight functions, Ojdrovic and Petroski [11] assumed the derivative of the crack profile to be in the form of a series:

$$\frac{\partial u(a,x)}{\partial a} = \frac{2\sigma}{H}\sqrt{2}\sum_{j=0}^{M} c_j \left(1 - \frac{x}{a}\right)^{j-1/2} \; ; \; c_0 = \frac{F\left(a/d\right)}{2}. \tag{6}$$

By knowing a number of reference stress intensity factor values known as Multiple Reference States, a number of the unknown coefficients can be derived. In other words, assuming that *M* stress intensity factors are known for a particular geometry under *M* symmetric loading states, from Eq. 4 and Eq. 5 the following can be derived:

$$\int_0^a H\sigma_i(x)\frac{\partial u_1(a,x)}{\partial a}dx = K_i(a)K_1(a)$$

Where i=1,…,M. By substituting Eq. 6 into this equation, a system of *M* equations with *M* unknown is formed which can be solved to give the coefficients in Eq. 6. Brennan [13] has given a more portable form of the Multiple Reference States method in the form of a matrix equation.

In his classic paper on weight functions [10], Rice points out that there are cases for which knowledge of an integrated average of the intensity factor is sufficient for the calculation of the weight function. However, for arbitrary loadings, the RMS stress intensity factor calculation for surface cracks is extremely complicated. With drawing analogy with Rice's work [10], Besuner [14] used the energy balance principle for an increment of the crack growth to calculate the weight function and derived the following expression for the average stress intensity factor:

$$\overline{K}_i = \left(\frac{2\partial\left[\iint A\sigma_{zz}{}^{*} q^{*} dA\right]}{H\partial A_i}\right)^{-\frac{1}{2}} \iint_A \sigma_{zz}\frac{\partial q^{*}}{\partial A_i} dA\,.$$

This equation requires that for a reference crack face loading ($\sigma_{zz}{}^{*}$), the corresponding crack face displacement field be known (q^{*}) [14]. However, since unlike the two-dimensional case, there are no exact solutions for the crack face displacement of surface cracks, accurate derivation of weight functions is problematic. This means that recourse has to be made on the existing SIF values for constructing the weight functions.

3. Derivation of weight functions for surface cracks

Fett [15] suggested a method for deriving an approximate RMS SIF weight function based on an approximation of the crack profile and a number of reference solutions. It is in direct analogy with the work of Ojdrovic and Petroski [11], though the unknown coefficients are derived in a somewhat more complicated manner. Without going into details, a few problems with this method of WF derivation are mentioned:

1) The weight functions are complicated functions and their numerical calculation is extremely cumbersome.
2) Comparison of the two and three term weight functions (i.e. using one and two reference solutions respectively) for the width direction (h_c) for the same crack shows a great difference between the two. For $\varphi = 90$, the two term weight function gives negative values for the weight function, which is not possible and shows an error which can not be neglected.
3) Studies show that the deviation between the two-term and three-term weight function becomes greater for larger aspect ratios. Fett [15] does not give any comparison between the two-term and three-term WFs for the surface direction.

Therefore a novel approach using the MRS technique for the evaluation of weight function in surface cracks is presented here. Starting from

$$H \int_S \sigma_n \frac{\partial u_0}{\partial(\Delta S)} dS = \frac{1}{\Delta S} \int_{\Delta S} K_0 K_n d(\Delta S) \tag{7}$$

And defining the average stress intensity factor as

$$\overline{K}_{nA} = \frac{1}{\overline{K}_{0A}} \frac{1}{\Delta S_A} \int_{\Delta S_A} K_0 K_n d(\Delta S_A) \tag{8}$$

And therefore for the reference case, the average SIF would be

$$\overline{K}_{0A}^{\,2} = \frac{1}{\Delta S_A} \int_{\Delta S_A} K_0^2(\phi) d(\Delta S_A).$$

Now by defining m_A as $m_A = \dfrac{H}{K_0} \dfrac{\partial u_0}{\partial(\Delta S_A)}$ Eq. 6 becomes

$$\overline{K}_{nA} = \frac{1}{\overline{K}_{0A}} \int_S \sigma_n m_A K_0 dS$$

for the 'A' direction, and the same could be derived for the 'B', or surface' direction. So again from Eq. 6, for the 'A' direction, it follows:

$$\frac{2H}{\pi c} \int_S \sigma_n \frac{\partial u_0}{\partial(\Delta a)} dS = \frac{2}{\pi c \Delta a} \int_{\Delta S_A} K_0 K_n d(\Delta S_A)$$

where the following geometric relations can be derived:

$$d(\Delta S_A) = \Delta a_p dx\,,\quad \Delta a_P = \Delta a \sqrt{1 - \frac{x^2}{c^2}}\,,\ dS = dxdy$$

$$H\int_{-c}^{c}\int_{0}^{a}\sigma_n \frac{\partial u_0}{\partial(\Delta a)}dydx = \int_{-c}^{c} K_0 K_n \sqrt{1-\frac{x^2}{c^2}}dx . \tag{9}$$

So far no assumption has been introduced. Now if the derivative of the crack face displacement is assumed to be approximately expressed by the following finite series

$$\left.\frac{\partial u_0}{\partial(\Delta a)}\right|_{\varphi} = \frac{\sigma_0}{H}\sum_{j=0}^{m} C_j f(x,y)^{j-\frac{1}{2}} ,$$

then Eq. 7 can be written as

$$\int_{-c}^{c}\int_{0}^{a}\sigma_0\sigma_n \sum_{j=0}^{m} C_j f(x,y)^{j-\frac{1}{2}} dydx = \int_{-c}^{c} K_0 K_n \sqrt{1-\frac{x^2}{c^2}}dx .$$

Following the MRS methodology as introduced by Ojdrovic-Petroski [16] for the two-dimensional cracks, by letting

$$W_{ij} = \int_{-c}^{c}\int_{0}^{a}\sigma_0\sigma_i(x,y) f(x,y)^{j-\frac{1}{2}} dydx \text{, and } p_i = \int_{-c}^{c} K_0 K_i \sqrt{1-\frac{x^2}{c^2}}dx$$

and using m reference solutions, the following set of simultaneous equations is obtained

$$\sum_{j=0}^{m} W_{ij} C_j = p_i . \tag{10}$$

Based on an analogy to the two-dimensional weight function, the following functional form has been chosen for f:

$$f(x,y) = \left(1-\frac{y}{a}\right)\left(1-\frac{|x|}{c}\right).$$

Now if Newman and Raju solutions [ref] for the SIF are taken as reference values, Eq. 8 could be rewritten, for m=2, as

$$\sum_{j=0}^{1} W_{ij} C_j = p_i - W_{i2} C_2 = q_i$$

From which the unknown coefficients are derived as

$$C_0 = \frac{q_1 W_{21} - q_2 W_{11}}{W_{10}W_{21} - W_{20}W_{11}} \text{ and } C_1 = \frac{q_2 W_{10} - q_1 W_{20}}{W_{10}W_{21} - W_{20}W_{11}} .$$

Therefore the weight function is derived as

$$\frac{H}{\sigma_0}\frac{\partial u_0}{\partial(\Delta a)} = \sum_{j=0}^{m} C_j \left(1-\frac{|x|}{c}\right)^{j-\frac{1}{2}}\left(1-\frac{y}{a}\right)^{j-\frac{1}{2}} .$$

Now the stress intensity factor can be calculated, using the above weight function in the Cartesian coordinate system, as

$$\overline{K}_{nA} = \frac{2}{\pi c}\frac{1}{\overline{K}_{0A}}\int_S \sigma_0\sigma_n \sum_{j=0}^{m} C_j \left(1-\frac{|x|}{c}\right)^{j-\frac{1}{2}}\left(1-\frac{y}{a}\right)^{j-\frac{1}{2}} dS$$

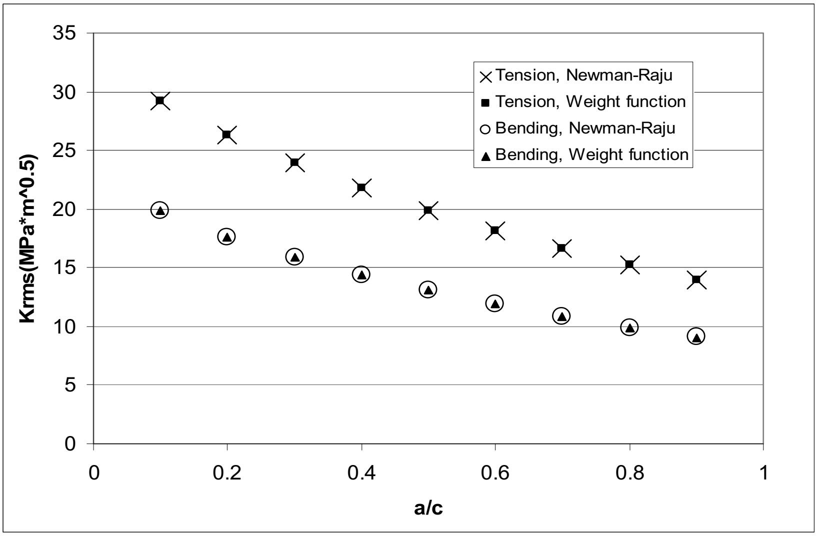
Where m is the number of reference solutions.

4. Verification of the Weight Function

A wide range of crack aspect ratios for the semi-elliptical surface crack were modelled using a three-dimensional Finite Element model. For the tensile and bending cases, the weight function results show a near-exact match with the computed FE RMS SIFs. This is expected as the tensile and bending cases have been used as references in constructing the weight function. For validation purposes, two more types of loading have been used, namely $\sigma = \sigma_0\left(\frac{a-y}{a}\right)^2$, denoted as loading 1, and $\sigma = \sigma_0\left(\frac{a-y}{a}\right)^3$, denoted as loading 2. Fig. 2 shows a comparison between the RMS SIF values obtained using the weight function and the FE results, for different surface cracks with a fixed a/t of 0.3. It is observed that the weight function result for tension and bending coincide exactly with Newman-Raju values. This is expected as Newman-Raju values are used for reference solutions. For the two other loading cases that have been shown, *i.e.* $\sigma = \sigma_0\left(\frac{a-y}{a}\right)^2$ and $\sigma = \sigma_0\left(\frac{a-y}{a}\right)^3$, a good agreement is observed between the weight function results and the results obtained from finite elements method.

5. Summary

This paper examined the growth of surface cracks and the use of RMS SIF values for life predictions using Paris law. The concept of Multiple Reference States for weight function derivation was discussed and a novel weight function was introduced which is far easier to apply than the existing weight functions for the semi-elliptical crack. Where an approximation was made in the process of WF derivation, this was done carefully and emphasis was made to point it out. For a range of crack aspect ratios, the values of the RMS SIF obtained from this WF were compared against the FE values for four different loading cases and the results showed great consistency.



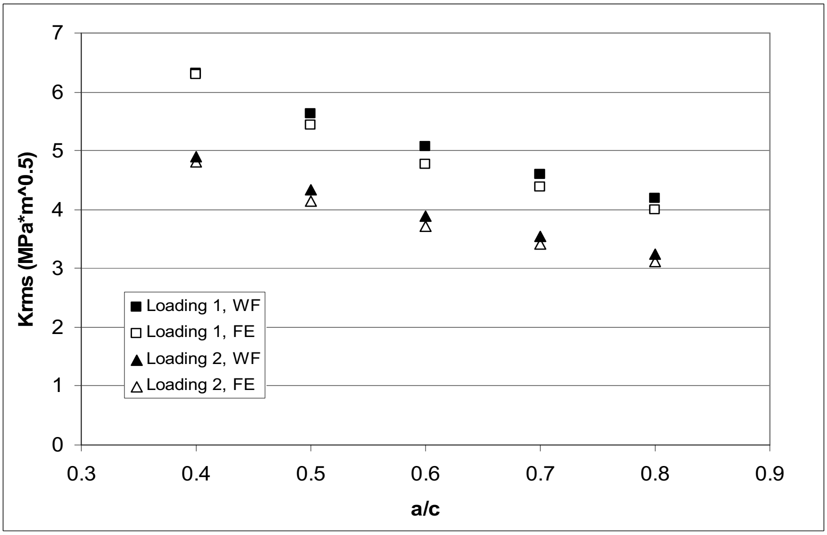


Fig. 2- Comparison between the weight function and the finite elements results for different loadings

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