Reliability Estimation for Structure under Fatigue Load Using Probability Theory and Fatigue Crack Growth Model

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Abstract. Methodologies to calculate the failure probability and to estimate the reliability of the damaged structures under fatigue are developed in the present work. The applicability of the developed methodologies is evaluated with the help of fatigue crack growth models suggested by Paris and Walker. Probability theories such as the FORM (first order reliability method) and the SORM (second order reliability method) are utilized for the seven cases. It is found that the failure probability increases with the increase of the initial fatigue crack size and the decrease of the design fatigue life. It is also found that the failure probability calculated by the FORM and the SORM turns out similar for the Paris and the Walker models. And the distribution types of the slope of the Paris equation and the coefficient of the Paris equation dominantly affect the failure probability.

Introduction

The repeated loads may lead to failure of material even when the load level is lower than the ultimate limit states. Many mechanical structures such as train axles and wheels, load bearing parts of automobiles, offshore structures, and bridges are designed to endure for a long term up to giga-cycle loadings in the actual service. Furthermore, some mechanical structures in the various areas are needed to be investigated if the operation life can be extended beyond the design life because of the economic consideration. In such circumstances, mechanical components of these structures are exposed to tremendous number of stress/strain cycles in the long term service. Thus, the fatigue property of the structural materials under the long term cyclic loadings is an important subject to provide the safety design data for such mechanical structures [1,2].

In the fatigue design, the use of S-N curves is well established. These curves predict fatigue failure under constant amplitude loading, but cannot incorporate information related to crack detection and/or measurement. As a result, the structures must be repaired, if the crack is discovered. However, the use of fracture mechanics techniques can be successfully applied to this problem. The fracture mechanics needs the information about the defects, or cracks to be used in the analysis. Since the size and location of defects are quite random, the deterministic analysis may provide incomplete results about the structure safety. Also the randomness of loads, geometry and material properties influence significantly the reliability of a structure. Therefore, the fracture mechanics with a probabilistic method provide a useful tool to solve these problems [3,4].

In this paper, fatigue models suggested by Paris and Walker are used to formulate the limit state function for assessing the failure of fatigue loaded structures. And the failure probability is estimated by using the FORM (first order reliability method) and the SORM (second order reliability method). The reliability is assessed by using this failure probability, and the application of these methods to the reliability estimation is given for a case study.

Fatigue Models

The strength of a component or structure can be significantly reduced by the presence of a crack. The fatigue crack growth rate, da/dN, versus the applied stress intensity factor range, ΔK can be obtained from fatigue crack propagation experiments. The corresponding applied stress intensity factor range, ΔK, is calculated when the crack length, a and the applied stress range, ΔS, are measured in the experiments as below [1,4-6].

$$\Delta K_I = \Delta K = K_{\max} - K_{\min} = S_{\max}\sqrt{\pi a}\alpha - S_{\min}\sqrt{\pi a}\alpha = (S_{\max} - S_{\min})\sqrt{\pi a}\alpha = \Delta S\sqrt{\pi a}\alpha \,. \tag{1}$$

Where α is the geometry factor. Since the stress intensity factor is undefined in the compression, $K_{\min}$ is taken as zero if $S_{\min}$ is compressive. The correlation for constant amplitude loading is usually a log-log plot of the fatigue crack growth rate, da/dN, in $m/cycle$, versus the opening mode stress intensity factor range, ΔK_I (or ΔK), in $MPa\sqrt{m}$.

The typical log-log plot of fatigue crack growth rate versus stress intensity factor range as shown schematically in Fig. 1 has a sigmoid shape that can be divided into three major regions. Region I is the near threshold region and indicates a threshold value, ΔK_{th}, and there is no observable crack growth below this value. This threshold occurs at crack growth rates on the order of $1\times10^{-10}\, m/cycle$ or less. Region II shows essentially a linear relationship between log da/dN and $\log\Delta K$, which corresponds to the formula suggested by Paris [1,3,5].

$$\frac{da}{dN} = C(\Delta K)^n \,. \tag{2}$$

Where n, C are material constants. n is the slope of the line and C is the coefficient found by extending the straight line to $\Delta K = 1 MPa\sqrt{m}$. Region II fatigue crack growth corresponds to stable macroscopic crack growth that is typically controlled by the environment. Microstructure and mean stress have less influence on fatigue crack growth behavior in region II than in region I. In region III, the fatigue crack growth rates are very high as it approaches instability, and little fatigue crack growth life is involved. This region is controlled primarily by fracture toughness K_C or K_{IC}, which depends on the microstructure, mean stress, and environment.

Conventional S-N or ε-N fatigue behavior is usually referenced to the fully reversed stress or strain conditions ($R=-1$). However, fatigue crack growth data are usually referenced to the pulsating tension condition with $R=0$ or approximately zero.

The general influence of mean stress on fatigue crack growth behavior can be estimated by using the stress ratio, $R = K_{\min}/K_{\max} = S_{\min}/S_{\max}$, which is used as the principal parameter and has the positive value, $R \ge 0$. It should be recognized that the effect of the R ratio on the fatigue crack growth behavior is strongly material dependent.

A common empirical relationship used to describe mean stress effects with $R \ge 0$ is the Walker equation as below [1,3-5].

$$\frac{da}{dN} = \frac{C(\Delta K)^n}{(1-R)^{n(1-\lambda)}} = C''(\Delta K)^n \,. \tag{3}$$

Where C and n are the coefficient and slope of Paris equation for $R=0$, respectively, and λ is a material constant. Paris equation and Walker equation are basically similar, with different coefficients of the equations, C and C'', as below.

$$C'' = \frac{C}{(1-R)^{n(1-\lambda)}} . \tag{4}$$

Because the effect of R on fatigue crack growth is known as material dependent, it is necessary to determine the material constant, λ. Value of λ for various metals ranges from 0.3 to nearly 1, with a typical value of around 0.5.

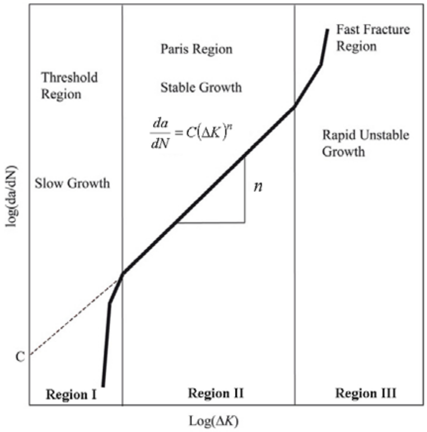

Fig. 1 Schematic behavior of fatigue crack growth rate versus stress intensity factor range.

Probability Theory

FORM (first order reliability method). The failure probability is calculated by using the FORM, which is one of the methods utilizing the reliability index. The FORM method is based on the first-order Taylor series approximation of a limit state function (LSF), which is defined as below [6-8].

$$Z = RE - LO . \tag{5}$$

Where, RE is the resistance normal variable, and LO is the load normal variable. Assuming that RE and LO are statistically independent, normally distributed random variables, the variable Z is also normally distributed. The failure occurs when $RE < LO$, i.e., $Z < 0$. The failure probability is given as below.

$$PF = P[Z<0] = \int_{-\infty}^{0} \frac{1}{\sigma_Z\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{Z-\mu_Z}{\sigma_Z}\right)^2\right\} dZ = \int_{-\infty}^{-\beta} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{U^2}{2}\right\} dU = \Phi(-\beta). \tag{6}$$

Where μ_Z and σ_Z are the mean and standard deviation of the variable Z, respectively, and Φ is the cumulative distribution function for a standard normal variable, and β is the safety index or reliability index and the coefficient of variation (C.O.V) denoted as below.

$$\beta = \frac{\mu_Z}{\sigma_Z} = \frac{\mu_R - \mu_L}{\sqrt{\sigma_R^2 + \sigma_L^2}} , \quad C.O.V = \frac{\sigma_X}{\mu_X} . \tag{7}$$

Equation (7) can be used when the system has a linear LSF. Actually, most real systems and cases do not have linear LSF but rather a nonlinear LSF. So, for a system that has a nonlinear LSF, Eq. (7) cannot be used to calculate the reliability index. Rackwitz and Fiessler proposed a method to estimate the reliability index that uses the procedure shown in Fig. 2 for a system having a nonlinear LSF. In this paper, we iterate the loop, as shown in Fig. 2, to determine a reliable reliability index until the reliability index converges to a desired value ($\Delta\beta \le 0.001$) [7,8].

The LSF must be defined to formulate the FORM and evaluate the reliability. In this paper, the LSF can be defined by using the fatigue models as below [6,7].

$$Z = N_D - N_f \,. \tag{8}$$

Where, N_D is the design fatigue life and N_f is the fatigue life estimated from the fatigue crack growth models such as Paris and Walker models using Eq. (2) or (3).

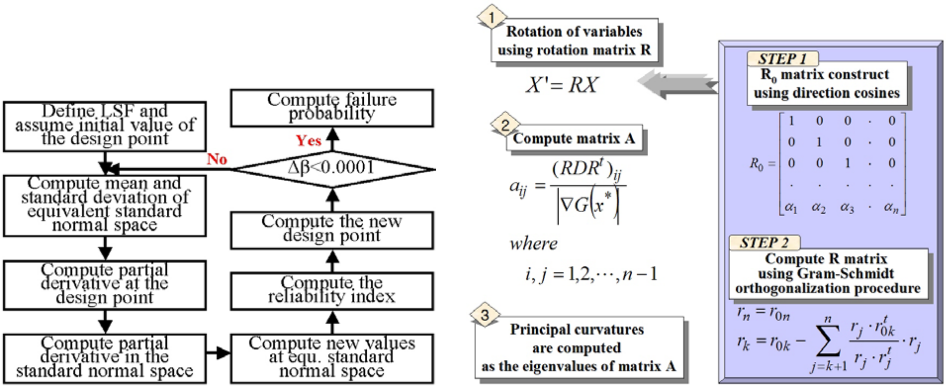

Fig. 2 Computation process of the reliability index and principle curvatures.

SORM (second order reliability method). The computations required for reliability analysis of systems with linear LSF are relatively simple. However, the LSF could be nonlinear either due to a nonlinear relationship the random variables in the LSF or due to some variables being non-normal.

The FORM approach will give the same reliability index for both linear and nonlinear limit state cases, if the minimum distance point is same. But it is apparent that the failure probability of the nonlinear limit state would be less than that of the linear limit state, due to the difference in the failure domains. The curvature of the limit state around the minimum distance point determines the accuracy of the first order approximation in the FORM. The SORM improves the FORM result by including additional information about the curvature of the limit state.

The SORM approach was first explored by Fiessler using various quadratic approximations. A simple closed form solution for probability computation using a second order approximation and adopting the theory of asymptotic approximation was given by Breitung [6-8].

$$PF_{SORM} = \Phi(-\beta)\prod_{i=1}^{n-1}(1-\beta\kappa_i)^{-1/2} \,. \tag{9}$$

Where κ_i denotes the principal curvatures of the LSF at the minimum distance point and β is the reliability index calculated by using the FORM. The principal curvatures are computed by using steps shown in Fig. 2.

Non-normal Distribution. The FORM initially assumed that all random variables have normal distribution. However, the random variables have non-normal distributions at the real situation. For the case of having non-normal distribution, Rackwitz and Fiessler proposed the transformation method using the equivalent normal distribution with the parameters of μ_X^N and by using equation defined like below [7,8].

$$F_X(x^*) = \Phi\left(\frac{x^* - \mu_X^N}{\sigma_X^N}\right), \quad \mu_X^N = x^* - \Phi^{-1}\left(F_X(x^*)\right)\sigma_X^N \; . \tag{10}$$

$$f_X(x^*) = \frac{1}{\sigma_X^N}\phi\left(\frac{x^* - \mu_X^N}{\sigma_X^N}\right), \quad \sigma_X^N = \frac{\phi\left[\Phi^{-1}\left(F_X(x^*)\right)\right]}{f_X(x^*)}. \tag{11}$$

The non-normal distributions applied are the lognormal and the Weibull distributions. The lognormal distribution is commonly used for general reliability analysis, cycle-to-failure in fatigue, material strength and loading variable in probabilistic design. λ and ς are the two parameters characterizing the lognormal distribution. The Weibull distribution is commonly used to describe material strengths and time to failure of electronic and mechanical devices and components. We take the two-parameter Weibull distribution. Where the shape parameter, m and the scale parameter, c are the two values in the two-parameter Weibull distribution. The PDF, the relationships among parameters, and the means and variances of the lognormal and Weibull distributions are presented in Table 1 [8].

Table 1 Characteristics of varying distributions.

Distribution	PDF	Parameter	Mean, Variance
Normal Distribution	$f_Z(x) = \frac{1}{\sigma_Z\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\frac{x-\mu_Z}{\sigma_Z}\right)^2\right]$	μ_Z, σ_Z	$E = \mu_Z$, $Var = \sigma_Z^2$
Lognormal Distribution	$f_Z(x) = \frac{1}{\varsigma x\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\frac{\ln x-\lambda}{\varsigma}\right)^2\right]$	λ, ς	$E = \exp\left(\lambda + \frac{1}{2}\varsigma^2\right)$, $Var = E^2\left(e^{\varsigma^2} - 1\right)$
Weibull Distribution	$f_Z(x) = \left(\frac{m}{c}\right)\left(\frac{x}{c}\right)^{m-1}\exp\left[-\left(\frac{x}{c}\right)^m\right]$	m, c	$E = c\Gamma\left(1+\frac{1}{m}\right)$, $Var = c^2\left[\Gamma\left(1+\frac{2}{m}\right) - \Gamma^2\left(1+\frac{1}{m}\right)\right]$

Table 2 Random variables and their statistical values used in the case study.

Valuable	Mean	C.O.V.
S_{max}	200 MPa (Paris)	0.002
	300 MPa (Walker)	0.002
S_{min}	-50 MPa (Paris)	0.002
	100 MPa (Walker)	0.002
a_i	0.001 m	0.01
C	6.9×10^{-12}	0.02
n	3.0	0.02
N_D	129,000 $cycle$ (Paris)	0.003
	65000 $cycle$ (Walker)	0.003

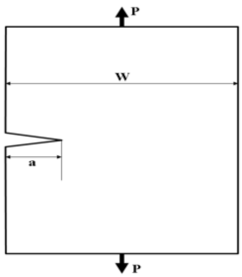

Fig. 3 The geometry of single edge crack specimen.

Table 3 Cases classified by the distribution types of random variables.

	Case 1	Case 2	Case 3	Case 4	Case 5	Case 6	Case 7
$S_{\max}$	Normal	Log-normal	Weibull	Weibull	Log-normal	Log-normal	Weibull
$S_{\min}$	Normal	Log-normal	Weibull	Weibull	Log-normal	Log-normal	Weibull
C	Normal	Log-normal	Weibull	Log-normal	Normal	Weibull	Log-normal
n	Normal	Log-normal	Weibull	Log-normal	Normal	Weibull	Log-normal
a_i	Normal	Log-normal	Weibull	Normal	Log-normal	Log-normal	Weibull
K_c	Normal	Log-normal	Weibull	Weibull	Weibull	Weibull	Weibull
N_D	Normal	Log-normal	Weibull	Normal	Log-normal	Log-normal	Weibull

A Case Study

In this paper, we formulate the LSF using the fatigue models, and the failure probability is estimated using the FORM and the SORM with the given data for the fatigue experiment in a single edge crack shown in Fig. 3. The specimen is a very wide SAE 1020 cold-rolled thin plate subjected to constant amplitude uni-axial cyclic loads. The random variables and their statistical values used in the fatigue models are listed in Table 2. In this paper, it is classified as seven cases according to the distribution types of random variables and the failure probability of the fatigue damaged structures is systematically studied and compared for the seven cases [1-5].

Results and Discussion

The distribution types of random variables are obtained from the references and the seven cases are classified as shown in Table 3. And the failure probability to assess the reliability of the fatigue damaged structures is calculated using the FORM and the SORM according to the data listed in Table 2. The limit state function to calculate the failure probability using the FORM and the SORM is formulated by using the Paris and Walker models.

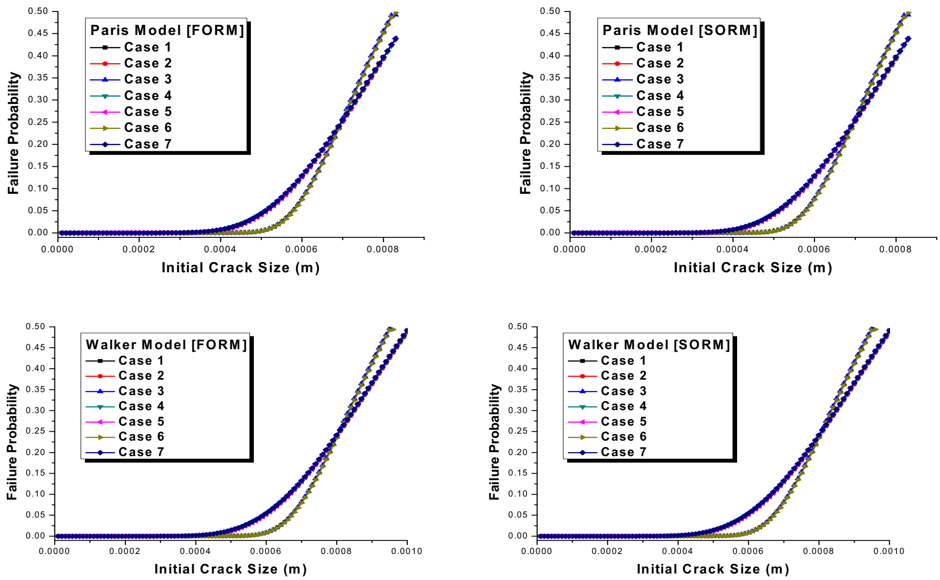

Fig. 4 A relationship between failure probability and initial crack size for Paris and Walker model according to FORM and SORM with respect to varying cases.

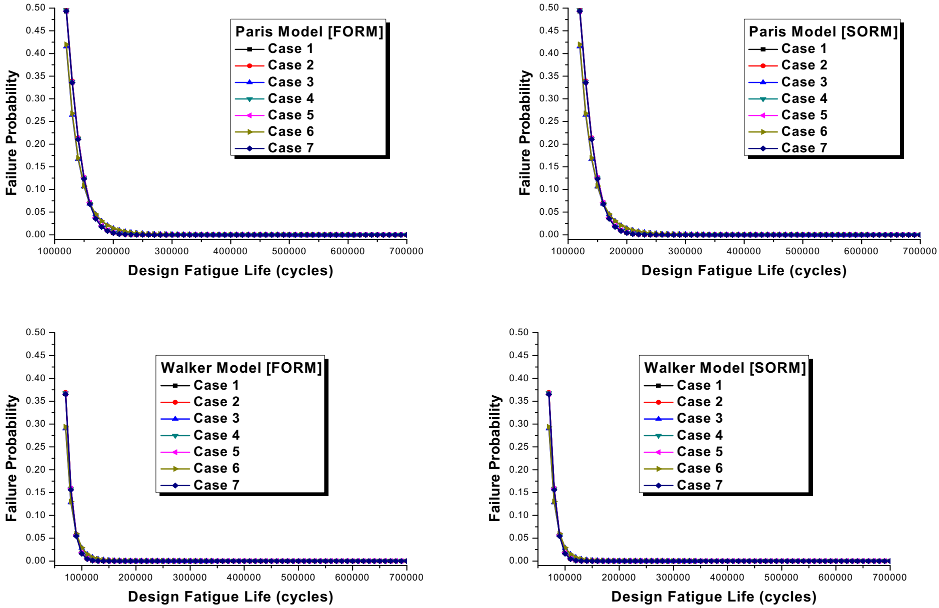

Fig. 5 A relationship between failure probability and design fatigue life for Paris and Walker model according to FORM and SORM with respect to varying cases.

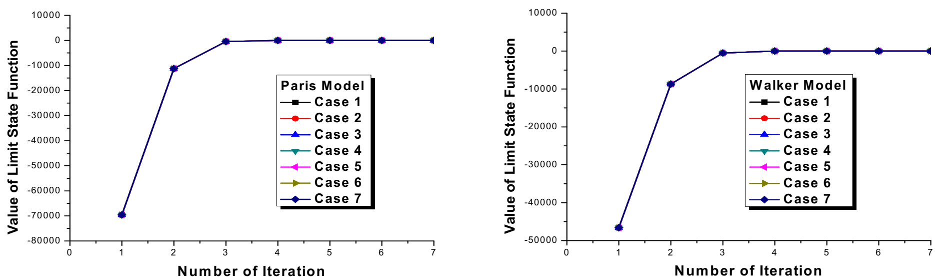

Fig. 6 Convergence phase in terms of the number of iteration in the Paris and the Walker models with seven cases.

It is recognized from Fig. 4 and Fig. 5 that the failure probability increases with increase of initial crack size and decrease of design fatigue life. And it is found from Figs. 4 and 5 that the Paris model shows slightly large failure probability than the Walker model. It is found from Figs. 4 and 5 that the failure probability of Case 3 and Case 6 is similar with varying initial crack size and design fatigue life. And it is found that the failure probability of Cases 3 and 6 increases steeply than other cases and becomes larger than other cases at more than about 0.0007m of initial crack size and less than 160,000 cycles of design fatigue life for the Paris model, and at more than about 0.008m of initial crack size and less than 80,000 cycles of design fatigue life for the Walker model.

It seems from Figs. 4 and 5 that the failure probability is similar, if all random variables have the normal and lognormal distributions. And it seems that the distribution types of all random variables except the slope of Paris equation and the coefficient of Paris equation doesn't affect on the failure probability. It seems that the distribution types of the slope of Paris equation and the coefficient of Paris equation are very important for determination of failure probability. That is, the failure probability is slightly different, if the slope of Paris equation and the coefficient of Paris equation show Weibull distribution.

In this paper, the failure probability calculated by using the probability theories such as the FORM and the SORM is used to analyze the reliability of the structure having the fatigue damage. The failure probability can be calculated by using the iteration method proposed by Rackwitz and Fiessler. The variation of the value of limit state function with the step of iteration is shown in Fig. 6. Although it is not expressed definitely in Fig. 6, it is found that the Walker model converges to the some specific value rapidly than Paris model and the value of limit state function is similar for all cases. And it is also found that limit state function has been converged less than three steps of iteration. Although it isn't shown in this paper, we verify that it shows similar tendency for the variation of other variables such as the design fatigue life, the maximum stress, the minimum stress and slope of the Paris equation.

Conclusions

In this paper, the fatigue crack growth models suggested by Paris and Walker are used to formulate the LSF. And the FORM and the SORM are used to estimate the failure probability for seven cases and the following results are obtained:

(1) It is found that the failure probability increases with the increase in the initial fatigue crack size and the decrease in the design fatigue life in the Paris and the Walker models.

(2) It is found that the FORM and the SORM show similar failure probability for the Paris and the Walker models.

(3) It is verified that in the Walker model it rapidly converges to a specific value than in the Paris model and the value of limit state function is similar for the case in which all of the cases.

Acknowledgments

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