Kinetic description of dynamic crack propagation

Arseny Kashtanov[1, a], Yuri Petrov[1,b] and Leonid Isakov[1,c]

[1]St.-Petersburg State University, Mathematics & Mechanics Faculty,

198504, Universitetsky pr., 28, St.-Petersburg, Russia

[a]kashto@mail.ru, [b]yp@YP1004.spb.edu, [c]IsakovLeonid@mail.ru

Keywords: kinetic fracture equation, fracture criterions, dynamic fracture, dynamic damage, fracture wave.

Abstract. An approach describing the dynamic fracture as a process of nucleation and subsequent propagation of a nonlinear wave of microfracture is justified. It operates with a function defining the level of material damage during incubation processes. The proposed approach provides the detailed investigation of dynamic and static fracture as well as the statement of strength criteria at the macroscale.

Introduction

Experimental investigations of the dynamic fracture of solids reveal the fundamental difference in medium response to the dynamic and static loading conditions. In contrast to the static case, where the experimentally determined critical characteristics of strength are the material constants, the dynamic-strength properties of materials are strongly unstable and dependent on the loading history and other factors. Typically, the dynamic-strength diagram is characterized by two asymptotes corresponding to extremely short and extremely long loading pulses. To describe this typical dynamic effect in fracture problems the concept of fracture incubation time was proposed by Morozov, Petrov and Utkin [1] based on the notion of characteristic time of micro-fracture and corresponding relaxation (incubation) processes preceding the macro-fracture event. Introducing the incubation time as a structural material parameter together with static strength and fracture toughness allows to state the macroscopic fracture criterion that turned out to be very powerful to describe experimentally observed effects of dynamic fracture. But a continual description of fracture evolution and corresponding incubation process at the microscopic scale level has not been provided.

To describe the microfracture evolution (including the processes of nucleation, interaction and following coalescence of microfracture – microcracks, microdamage, vacancies and so on) the new diffusive-temporal approach was proposed in [2] in joint authorship with N. Pugno and A. Carpinteri. Suggested approach introduces the function defining a damage level of material during the incubation processes to provide the detailed investigation of the preliminary stage of dynamic fracture event as well as the following propagation stage. Thus, the process of dynamic crack nucleation was simulated and compared with experimental results. But in that paper it was not been given the rigorous mathematical justification of dynamic damage equation. The basic equation based on quite common considerations was there partially postulated. Here we fill this gap and provide the consistent deduction of considered approach. It is based on incubation time criterion, mass conservation principles and transfer equation.

Firstly, we will provide the brief exposition of incubation time approach to clarify the main principles of structural fracture mechanics. Then, the new continual-temporal approach will be presented. It allows to describe the uniform process of microfracture evolution (damage accumulation process) in static and dynamic cases. It will be shown that it gives the same fracture criterion as incubation time approach and coincides in limit case with well-known quasistatic criteria and equations used in fracture mechanics. Finally, the diffusion-temporal description of fracture process will be obtained based on transfer equation, and methods for definition of its parameters will be outlined.

Authors would like to thank Prof. Alberto Carpinteri and especially Prof. Nicola Pugno for fruitful discussions and helpful comments.

Structural-Temporal Criterion

Nowadays, the nonoperability of traditional quasistatic fracture models in the case when the fracture happens in rather short time intervals after the beginning of exterior pulse application (which corresponds to high loading rates) became apparent. The dynamic fracture is accompanied by high local deformation velocities and medium particles, adjacent to the rupture place, can move extremely fast. Thus, at the modeling of dynamic fracture mechanism one should consider an inertia effect together with elastic resistance of the material. The structural-temporal approach *integrally* considering this phenomenon was proposed by Morozov, Petrov and Utkin [1]. Here we would like to present the brief exposition of its main statements (more sequential description one can find in [3,4]).

For simplicity let us consider some arbitrary two-dimensional stress field (e.g. the plane strain) and suppose that fracture will happen along some direction Ox being the symmetry line. Referring to the classical force approach of fracture mechanics, in the case of *quasistatic* loading a fracture occurs when the instantaneous local force acting in the supposed place of rupture attains a critical value

$$F \leq F_C. \tag{1}$$

In the terms of continual stress field σ the criterion (1) can be written in the form

$$\int_{x-d}^{x} \sigma(x')dx' \leq \sigma_* d, \tag{2}$$

where d is some still undefined linear size describing the spatial structure of solid and σ_* could be considered as some critical stress introduced instead of critical force F_C. The criterion (2) is known as *Neuber-Novozhilov criterion*. The basic principles of Neuber-Novozhilov's approach could be reduced to the following statements (similar to the basic principles of quantum mechanics):

1. all solids consists of spatial-structural elements of finite size;
2. an elementary act of fracture is a fracture of one structural element;
3. criterion parameters, including a structural element dimension, should be chosen to preserve the results of classical fracture theory in the limit of low load rates.

Considering the case of intact fracture and corresponding fracture criterion of critical stress $\sigma \leq \sigma_C$ (where σ_C is the static strength of the material) and applying the third basic principle one will obtain

$$\sigma_* = \sigma_C. \tag{3}$$

On the other hand, let us consider the classic Griffith crack problem. Substituting the tensile stress in the crack tip $x = 0$

$$\sigma(x) = \frac{K_I}{\sqrt{2\pi x}} + O(1), \quad x \to 0 \tag{4}$$

into criterion (2) we obtain

$$\int_0^d \frac{K_I dx'}{\sqrt{2\pi x'}} = \sqrt{\frac{2d}{\pi}} K_I \leq \sigma_c d, \tag{5}$$

Corresponding Irwin fracture criterion is $K_I \leq K_{IC}$ (where K_I is the stress intensity factor and K_{IC} is the static fracture toughness) and then

$$d = \frac{2}{\pi}\frac{K_{Ic}^2}{\sigma_c^2}. \tag{6}$$

Criterion (2), in combination with (3) and (6), permits efficacious fracture forecast in many non-classical situations, e.g. the problem of crack appearing in the vertex of angular notch [5].

In the case of *dynamic fracture*, a high mobility of small but extremely fast medium particles determines the dependence not only on instantaneous components of the force field but also on the time of their action. And the criterion of fracture could be formulated in the following manner: the force pulse acting during some time period attains its critical value

$$J(t) \leq J_C, \quad J(t) = \int_{t-\tau}^{t}\int_{x-d}^{x} \sigma(x', t')dx'\,dt', \quad J_C = \sigma_C \tau d. \tag{7}$$

Here τ is the minimal time of load action required to cause the fracture by stress equal to σ_C, and d is defined by relation (6). The main peculiarity of *structural-temporal criterion* (7) is an introduction of some structural spatial d and temporal τ scales in explicit form. About the nature of temporal fracture scale τ called *incubation time* one can read in [3,4], here we will just notice that the structural scales d and τ can be considered as *macroscopic material constants*. Besides that the triple of experimentally defined material parameters σ_C, K_{IC} and τ defines totally the dynamic (and, obviously, static) fracture toughness and strength of material (see, e.g., [3,4]).

In the case, when the stress field does not depend on time, the structural-temporal criterion (7) coincides with Neuber-Novozhilov criterion (2,3,6). In addition, when the stress field near supposed rupture point depends only on time (e.g., in spalling fracture) the criterion (7) turns into

$$\frac{1}{\tau}\int_{t-\tau}^{t} \sigma(t')dt' \leq \sigma_C. \tag{8}$$

On the other hand in the case of macrocrack presence, resequencing steps (4)-(6) one obtain

$$\frac{1}{\tau}\int_{t-\tau}^{t} K_I(t')dt' \leq K_{IC}. \tag{9}$$

Criterions (8) and (9) can be rewritten in the following unified form (known as *incubation time criterion*)

$$\frac{1}{\tau}\int_{t-\tau}^{t} G(t')dt' \leq G_C, \tag{10}$$

where $G(t)$ is the local intensity of stress field and G_C represents its critical value. In the crack problems $G(t)$ coincides with the stress intensity factor $G(t) = K_I(t)$ and $G_C = K_{IC}$ is the static fracture toughness. At fracture of intact solids $G(t)$ is regarded as stress field $G(t) = \sigma(t)$ and $G_C = \sigma_C$ is the static strength.

Numerous investigations shown that incubation time criterion turned out to be very powerful to catch experimentally observed effects of dynamic fracture including dynamic branch of fracture toughness and temporal dependence of strength [3,4]. The static branches of those temporal dependences correspond to τ tending to zero (when the fracture criterion (10) degenerates into classical critical stress or Irwin's critical fracture toughness criteria). Therefore, *the incubation time criterion is the generalization of classical quasistatic approaches on the case of dynamic fracture*.

Finally, we have to note that the incubation time τ introduced in (6), as the minimal time of load action required to cause the fracture by stress equal to σ_C, has the physical meaning of *characteristic time of relaxation (microfracture) processes* accompanying the macrofracture in solids [3]. It reveals the fundamental role played by incubation time regarding to dynamic fracture processes.

Continual-Temporal Approach

The incubation time criterion allows an integral accounting of relaxation processes but a continual description of fracture evolution and corresponding incubation process at the microscopic scale level has not been provided. Here we would like to present the new *kinetic description of dynamic fracture* based on incubation time approach. It operates with a function corresponding to instant local microfracture state (*the damage function*) to describe the microfracture evolution (including the processes of nucleation, interaction and following coalescence of microfracture – microcracks, microdamage, vacancies and so on).

Let us consider a *spatially isotropic* process of microfracture evolution and fix an arbitrary small solid volume. Its mass is denoted as m, its volume before deformation is V_0, whereas the total volume of microfracture (damage) accumulated inside the chosen portion is V_*. Thus, during the damage accumulation process its volume changes as $V = V_0 + V_*$. The change of volume is obviously accompanied by a variation of local density $\rho = \rho(t)$, described by the mass conservation law

$$\frac{1}{\rho}\frac{d\rho}{dt} = -\operatorname{div}\overline{v}. \tag{11}$$

Here $\overline{v}$ is a local velocity of material particles. We can express the local density as $\rho = \frac{dm}{dV} = \frac{dm}{dV_0}\frac{dV_0}{dV} = \frac{dm}{dV_0}\left(1 - \frac{dV_*}{dV}\right)$. Introducing the damage parameter $\theta = \frac{dV_*}{dV}$ and setting $\rho_0 = \frac{dm}{dV_0}$ we obtain $\rho = \rho_0(1-\theta)$. Substituting this expression into eq. (11) yields

$$\frac{d\theta}{dt} = (1-\theta)\operatorname{div}\overline{v}. \tag{12}$$

Obtained equation has the form of kinetic equation describing the creepage. Its right part represents the source of microfracture and, then, it has to depend on time indirectly, through the local force field and current damage level. It is natural to believe that fracture intensifies with increasing of damage level (i.e. $\operatorname{div}\overline{v} \sim \theta^{\alpha}$) and increasing of local stress (we will consider $\operatorname{div}\overline{v} \sim \big(G(t) - G(t-\tau)\big)^{\beta}$, accounting the local force field changing during incubation period). From dimensional analysis we will finally have

$$\frac{d\theta}{dt} = \frac{C}{\tau}\theta^{\alpha}(1-\theta)\left(\frac{G(t)-G(t-\tau)}{G_C}\right)^{\beta}. \tag{13}$$

Here $C, \alpha, \beta \geq 0)$ are some dimensionless parameters which will be defined to obtain the known criteria in particular cases. Thus, considering $\tau \approx 0$ (i.e. $G(t) - G(t-\tau) \approx \tau\, dG/dt$) we obtain

$$\frac{d\theta}{dt} = \frac{C}{\tau}\theta^{\alpha}(1-\theta)\left(\frac{\tau}{G_C}\frac{dG}{dt}\right)^{\beta}. \tag{14}$$

To avoid an unlimited growth of damage rate when $\tau \to 0$ we have to demand $\beta = 1$. Then

$$\frac{d\theta}{\theta^{\alpha}(1-\theta)} = C\frac{dG}{G_C}. \tag{15}$$

We will consider the material having some initial damage $\theta(0) = \theta_0$, if the time starts in the moment of loading application, and state the criterion of macrofracture in the form $\theta(t_*) = \theta_*$, if t_* is the time to fracture. Here θ_0 and θ_* are experimentally defined parameters. Besides that, in the case $\tau \to 0$ we have to obtain the classical quasistatic strength criterion $G(t_*) = G_C$. Then, integrating (15) yields

$$C = \int_{\theta_0}^{\theta_*} \frac{d\theta'}{\theta'^{\alpha}(1-\theta')}. \tag{16}$$

The only possibility for constant C to be finite (at least for $0 < \theta_0 < \theta_* < 1$) is the case when $\alpha = 1$ and then

$$C = \ln\frac{\theta_*(1-\theta_0)}{\theta_0(1-\theta_*)}. \tag{17}$$

Finally, the kinetic equation describing the process of dynamic microfracture evolution into macroscopic fracture surface will have the form

$$\frac{d\theta}{dt} = \frac{C}{G_C\tau}\big(G(t) - G(t-\tau)\big)\theta(1-\theta), \tag{18}$$

where constant C is defined by (17). Let us note that equation (18) gives exactly the incubation time criterion of fracture. Indeed, rewriting it in the form

$$\frac{1}{\tau}\big(G(t) - G(t-\tau)\big) = \frac{G_C}{C}\frac{d\theta}{\theta(1-\theta)}, \tag{19}$$

integrating on $[0,t]$ (where $t \leq t_*$), and taking into account $C = \max_{0\leq t\leq t_*}\int_{\theta_0}^{\theta(t)}\frac{d\theta'}{\theta'(1-\theta')}$ one obtain the relation (10). By the way we can derive the explicit solution of eq. (19) as

$$\theta(t) = \frac{\theta_0 G_1(t)}{1-\theta_0+\theta_0 G_1(t)}, \quad \text{where } G_1(t) = \exp\left(\frac{C}{G_C\tau}\int_{t-\tau}^{t} G(t')dt'\right). \tag{20}$$

The *continual-temporal approach*, and particularly expression (20), is expected to be very fruitful in description of microfracture evolution during dynamic fracture process. At least, using very similar approach it was estimated the influence of suppressed stress state foregoing the spalling fracture [6].

In conclusion, we would like to mention that eq. (14) is the generalization of classical damage equation [7]. Considering one-dimensional tension of isotropic bar by constant load P applied on its ends we can express the stress in each bar's point as $G \equiv \sigma = P/(1-\theta)$. Then, in particular case $\alpha = 0$ and $\beta > 1$ eq. (14) yields

$$\frac{d\theta}{dt} = \frac{AP^k}{(1-\theta)P^{k-1}}, \quad \text{where } A = \frac{1}{\tau}\left(\frac{C}{G_C^{\beta}}\right)^{\frac{1}{1-\beta}} \quad \text{and } k = \frac{\beta}{1-\beta}. \tag{21}$$

Let us to emphasize one more time that obtained approach has the form of kinetic equation and, being the generalization of classical damage equation, gives in limit cases known fracture criteria. It is very important fact allowing to stop the "eternal argument" between supporters of kinetical and critical approaches. In fact, both of them could be obtained on common basis as it was shown above.

Diffusion-Temporal Approach

Till this moment we have considered spatially isotropic process of microfracture accumulation neglecting the redistribution of microfracture. To complete our model and *account the nonuniformity of damage evolution* we are going to derive the model describing the microfracture accumulation process in the form of *transfer equation*. We fix an arbitrary stationary domain Ω inside the considered solid and will, as previous, consider the damage function $\theta(x,t) \in [0,1]$ to characterize the relative volume of microfracture (microdamage) in solid's mass unit in the neighborhood of every point $x \in \Omega$. Then $\Theta_\Omega = \int_\Omega \rho_0(x)\theta(x,t)dx$ describes the evolution of local material density in Ω during the microfracture processes, where ρ_0 is the local density of the initial intact material. We can apply the transfer principle for Θ_Ω: the change of Θ_Ω inside of Ω is caused by a flux of microfracture through the boundary $\partial\Omega$ and by internal sources of microfracture. Owing to the fact that the domain Ω is fixed in space and due to its arbitrary choice we have

$$\rho_0 \frac{\partial \theta}{\partial t} + \nabla \cdot \bar{j}_\theta = s_\theta, \tag{22}$$

where $\bar{j}_\theta$ denotes the elementary flux of θ through the boundary $\partial\Omega$ and s_θ defines the specific rate of internal sources of θ inside Ω. Supposing the flux of microfracture over the boundary $\partial\Omega$ to be totally determined by diffusion-type processes of microfracture redistribution we can use the Fick's law $\bar{j}_\theta = -D\nabla(\rho_0\theta)$. The constant parameter D might be termed the relaxation factor. It has the physical meaning of the characteristic rate (intensity) of relaxation processes at the microscale.

Further we will not go beyond the one-dimensional case suitable for considering both major problems of dynamic fracture: crack propagation and spalling fracture. Neglecting the variation of density of an undamaged part of solid, eq. (22) is reduced to

$$\frac{\partial \theta}{\partial t} = D\frac{\partial^2 \theta}{\partial x^2} + S(\theta, x, t), \quad \text{where } S(\theta, x, t) = \frac{s_\theta}{\rho_0}. \tag{23}$$

Eq. (23) describes the microfracture evolution in the form of *diffusion equation*. This equation involves two functions, namely the relaxation factor D and the microfracture source function S. The expression for source function was derived above (the right part of eq. (18)); the meaning of relaxation factor with reference to the fracture process will be discussed later.

Now, let reduce eq. (23) to dimensionless form. Introducing new dimensionless variables $X = x/\sqrt{\tau D}$ and $T = t/\tau$, and dimensionless damage source intensity $\Phi(G) = C\left(G(T) - G(T-1)\right)/G_C$ we obtain

$$\frac{\partial\theta}{\partial T} = \frac{\partial^2\theta}{\partial X^2} + \Phi(G)\theta(1-\theta). \tag{24}$$

In fact, eq. (24) cannot be analytically integrated but if during some time period (longer than incubation time τ) the force field near the rupture point increases at constant velocity it coincides with well-known Kolmogorov-Petrovsky-Piskunov equation [8]:

$$\frac{\partial\theta}{\partial T} = \frac{\partial^2\theta}{\partial X^2} + \alpha\theta(1-\theta), \quad \text{where } \alpha = C\frac{G(T) - G(T-1)}{G_C} = \text{const} > 0. \tag{25}$$

This remarkable equation is invariant with respect to translation by X and T. That is, its admits *kink-type autowave* solutions having the form $\theta = \theta(X - \lambda T)$, where λ is some positive constant (autowave front velocity) defined by boundary conditions [8]. From the positions of crack mechanics it means that obtained equation could describe the self-sustaining crack propagation, when after initial loading pulse the crack goes to stable propagation mode drawing the energy from rupture of elastic bonds.

Another significant peculiarity of eq. (24) is an asymptotic behavior of its solutions. It is easy to see that $\theta = 0$ and $\theta = 1$ are theirs lower and upper asymptotes. Indeed, if $\theta(X_0, T_0)$ comes to 0 or 1 then eq. (24) becomes purely diffusive $\theta_T(X_0, T_0) = \theta_{XX}(X_0, T_0)$. In fact, it could be foretold already on the step of eq. (17) derivation. It was the reason of selecting the values θ_0 and θ_* different from 0 and 1, respectively.

Now, let us discuss the determination of relaxation factor D in practical applications. There is the direct method of its estimation having the numerical solution of eq. (24) and experimental data on macrocrack velocity $v(t)$. Indeed, let $V(T)$ be a numerically defined velocity of propagation of microfracture front. Coming back to the dimension variables we obtain

$$v(t) = \frac{dx}{dt} = \frac{dX}{dT}\sqrt{\frac{D}{\tau}} = V(T)\sqrt{\frac{D}{\tau}} \quad \text{and then } D = \tau\left(\frac{v(t)}{V(T)}\right)^2. \tag{26}$$

The expression (26) reveals the expected behavior of relaxation factor D: microcracks mobility decreases with decreasing of incubation time. That is, materials which are less effective in stress dissipation will fracture at smaller minimal times (at lower loads).

In such a way, we have finished the rigorous justification of *diffusive-temporal approach* (24) proposed in [2]. In conclusion we would like to note that this approach has been successfully used to foretell and simulate the dynamic crack nucleation process [2].

Summary

The diffusion-temporal approach (earlier suggested in [2]) was rigorously justified. On the basis of incubation time approach and principle of mass conservation the kinetic equation describing the uniform damage process in quasistatic and dynamic cases was derived. It was shown that corresponding continual-temporal approach, being the differential form of incubation time criterion, gives the classic damage equation in the limit quasistatic case and ensures the transition to well-known criteria used in fracture mechanics. Basing on transfer equation this approach was

generalized for the case of nonuniform damage accumulation process. Obtained equation describes dynamic fracture evolution as a process of nucleation and subsequent coalesce of microfracture. Particularly, it describes the propagation of macrocrack as a nonlinear microfracture wave. In the case of uniformly increasing stress field (or stress intensity factor) it can be reduced to the well-known Kolmogorov-Petrovsky-Piskunov equation. That is the obtained model gives the description of self-sustaining fracture as well. The application of presented approach to simulation of dynamic crack nucleation process was shown in [2].

Authors are obliged to the President of Russian Federation (grants НШ-2405.2008.1 and МК-3528.2008.1) for financial support.

References

[1] N.F. Morozov, Y.V. Petrov, A.A. Utkin: Doklady Akademii Nauk USSR. Vol. 313 (1990), p. 276–279 (in Russian).

[2] A.V. Kashtanov, Y.V. Petrov, N. Pugno, A. Carpinteri: submitted to Int. J. Fract. (2008). Manuscript Number FRAC951.

[3] N.F. Morozov, Y.V. Petrov: *Dynamics of Fracture*. Springer-Verlag, Berlin-Heidelberg (2000).

[4] N.F. Morozov, Y.V. Petrov, V.I. Smirnov: Fatig. Fract. Eng. Mater. Struct. Vol. 26 (2003), p. 363–372.

[5] N.F. Morozov: *Mathematical Problems of Crack Theory*. Nauka, Moscow (1994) (in Russian).

[6] A.V. Kashtanov, Y.V. Petrov: Doklady Physics. Vol. 52 (2007), p. 270–273.

[7] *Mechanisms of creepage and durability. Handbook*, edited by S.A. Shesterikov. Mashinostroenie, Moscow (1983) (in Russian).

[8] A.N. Kolmogorov, I.G. Petrovsky, N.S. Piskunov: Bull. Univ. d'Etat Moscou, Série internationale. Vol. A 1 (1937), P. 1–26 (in French).