From Nonlocal Damage to Fracture

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Keywords: damage, strain localization, softening, bifurcation, nonlocal continuum.

Abstract. Failure of quasibrittle materials such as concrete is characterized by gradual transition from diffuse microcracking to a localized damage process zone and finally to a macroscopic crack. Diffuse damage and its localization into a narrow band of a nonzero thickness are properly described by regularized constitutive models with softening, e.g. by integral-type nonlocal damage models. In this paper, a new nonlocal formulation of the isotropic damage model is proposed, with the damage-driving internal variable dependent on both the local and nonlocal equivalent strain. Localization properties of this model are studied analytically and numerically. It is shown that the localized process zone has initially a certain thickness controled by the nonlocal interaction radius, but later it shrinks and approaches zero thickness. The damage variable grows to 1 only in a very narrow band, which is an improvement over the standard formulation that leads to unrealistically wide bands of fully damaged material. For this reason, the new formulation is more appropriate for the description of transition to a localized crack.

Introduction

Realistic description of the mechanical behavior of quasibrittle materials such as concrete requires constitutive laws with softening. The physical origin of softening is in the propagation and coalescence of defects such as voids or cracks. It is well known that softening may lead to localization of inelastic strain into narrow process zones. For traditional models formulated within the classical framework of continuum mechanics, such zones have an arbitrarily small thickness, and failure can occur at arbitrarily small energy dissipation, which is not realistic. The boundary value problem becomes ill-posed and the numerical solutions suffer by pathological sensitivity to the discretization parameter, e.g. to the size of finite elements. To restore well-posedness, special enhancements acting as localization limiters must be used.

A wide class of efficient localization limiters is based on the concept of a nonlocal continuum. One popular nonlocal damage model uses the weighted spatial average of the equivalent strain as the damage-driving variable. The purpose of this paper is to propose a more general formulation and investigate its localization properties.

Local isotropic damage model

In this paper, we consider the class of isotropic damage models with one scalar damage variable ω, described by the stress-strain law

$$\boldsymbol{\sigma} = (1-\omega)\boldsymbol{D}_e : \boldsymbol{\varepsilon} \tag{1}$$

damage law

$$\omega = g(\kappa) \tag{2}$$

and loading-unloading conditions

$$f(\boldsymbol{\varepsilon}, \kappa) \equiv \varepsilon_{\rm eq}(\boldsymbol{\varepsilon}) - \kappa \leq 0, \qquad \dot{\kappa} \geq 0, \qquad f(\boldsymbol{\varepsilon}, \kappa)\,\dot{\kappa} = 0 \tag{3}$$

in which f is the damage loading function, g is the damage evolution function, $\varepsilon_{\rm eq}$ is a scalar measure of the strain level called the *equivalent strain*, and κ is an internal variable that corresponds to the maximum level of equivalent strain ever reached in the previous history of the material. The choice of the specific expression for the equivalent strain directly affects the shape of the elastic domain in the strain space, therefore it plays a role similar to the yield function in plasticity, which defines the yield surface in the stress space.

The shape of the stress-strain diagram is controled by the damage law (2). For instance, a model with linear elastic behavior up to the peak stress, followed by an exponential softening, is obtained with the following specific form of the damage law:

$$\omega = g(\kappa) = 1 - \frac{\varepsilon_0}{\kappa} \exp\left(-\frac{\langle \kappa - \varepsilon_0 \rangle}{\varepsilon_{\rm f} - \varepsilon_0}\right) \tag{4}$$

Here, ε_0 is the limit elastic strain, $\varepsilon_{\rm f}$ is a parameter that controls the steepness of the softening branch and thus the density of dissipated energy, and the Macauley brackets $\langle \ldots \rangle$ denote the positive part, i.e., $\langle x \rangle = x$ for $x > 0$ and $\langle x \rangle = 0$ otherwise.

From the rate form of the basic equations it is easy to derive the (elastic-damaged) tangent stiffness tensor

$$\boldsymbol{D}_{ed} = (1-\omega)\boldsymbol{D}_{\rm e} - g'\bar{\boldsymbol{\sigma}} \otimes \boldsymbol{\eta} \tag{5}$$

Here, $(1-\omega)\boldsymbol{D}_{\rm e}$ is the unloading stiffness, $g' = {\rm d}g/{\rm d}\kappa$ is the derivative of the damage function g, $\bar{\boldsymbol{\sigma}} = \boldsymbol{D}_{\rm e} : \boldsymbol{\varepsilon}$ is the effective stress, and $\boldsymbol{\eta} = \partial\varepsilon_{\rm eq}/\partial\boldsymbol{\varepsilon}$ is a second order tensor obtained by differentiation of the expression for equivalent strain with respect to the strain tensor. It is well known that if the corresponding localization tensor

$$\boldsymbol{Q}_{\rm ed} = \boldsymbol{n} \cdot \boldsymbol{D}_{ed} \cdot \boldsymbol{n} = (1-\omega)\boldsymbol{Q}_{\rm e} - g'(\boldsymbol{n} \cdot \bar{\boldsymbol{\sigma}}) \otimes (\boldsymbol{\eta} \cdot \boldsymbol{n}) \tag{6}$$

becomes singular for a certain direction characterized by a unit vector $\boldsymbol{n}$, then a weak discontinuity may develop across a plane perpendicular to $\boldsymbol{n}$. From the mathematical point of view, this correponds to the loss of ellipticity of the differential equation governing the deformable body, and the corresponding boundary value problem becomes ill-posed.

Damage model with nonlocal equivalent strain

A remedy to the problems mentioned at the end of the previous section is often sought in enriched formulations that serve as localization limiters. Typical examples include higher-order continua or integral-type nonlocal formulations [1]. One efficient and widely popular localization limiter is based on weighted spatial averaging of the equivalent strain. In the definition of the loading function, the local equivalent strain $\varepsilon_{\rm eq}$ evaluated at the given point is replaced by its nonlocal counterpart $\bar{\varepsilon}_{\rm eq}$, evaluated as a weighted average over a certain neighborhood of that point. Mathematically, the nonlocal equivalent strain at point $\boldsymbol{x}$ is defined as

$$\bar{\varepsilon}_{\rm eq}(\boldsymbol{x}) = \int_V \alpha(\boldsymbol{x}, \boldsymbol{\xi})\varepsilon_{\rm eq}(\boldsymbol{\varepsilon}(\boldsymbol{\xi}))\,{\rm d}\boldsymbol{\xi} \tag{7}$$

where $\alpha(\boldsymbol{x}, \boldsymbol{\xi})$ is a given *nonlocal weight function*. In an infinite body, the weight function depends only on the distance between the “source” point, $\boldsymbol{\xi}$, and the “receiver” point, $\boldsymbol{x}$. In the vicinity of a boundary, the weight function is usually rescaled such that the nonlocal operator does not alter a uniform field. This can be achieved by setting

$$\alpha(\boldsymbol{x},\boldsymbol{\xi}) = \frac{\alpha_0(\|\boldsymbol{x}-\boldsymbol{\xi}\|)}{\int_V \alpha_0(\|\boldsymbol{x}-\boldsymbol{\zeta}\|)\,\mathrm{d}\boldsymbol{\zeta}} \tag{8}$$

where $\alpha_0(r)$ is a monotonically decreasing nonnegative function of the distance $r = \|\boldsymbol{x}-\boldsymbol{\xi}\|$. In the one-dimensional setting, x and ξ are scalars and the domain of integration V reduces to an interval.

The weight function is often taken as the Gauss distribution function $\alpha_0(r) = \exp(-r^2/2\ell^2)$ where ℓ is a parameter reflecting the *internal length* of the nonlocal continuum. Another possible choice, adopted in the present study, is the truncated quartic polynomial function

$$\alpha_0(r) = \left\langle 1 - \frac{r^2}{R^2}\right\rangle^2 \tag{9}$$

where R is a parameter related to the internal length. Since R corresponds to the largest distance of point $\boldsymbol{\xi}$ that affects the nonlocal average at point $\boldsymbol{x}$, it is called the *interaction radius*. The Gauss function has an unbounded support, i.e., its interaction radius is infinite.

According to the modified loading-unloading conditions

$$\bar{\varepsilon}_{\mathrm{eq}} - \kappa \le 0, \qquad \dot{\kappa} \ge 0, \qquad (\bar{\varepsilon}_{\mathrm{eq}} - \kappa)\,\dot{\kappa} = 0 \tag{10}$$

the internal variable κ has the meaning of the largest previously reached value of *nonlocal* equivalent strain $\bar{\varepsilon}_{\mathrm{eq}}$. The corresponding damage variable evaluated from (2) is then substituted into the stress-strain equations (1). It is important to note that the damage variable is evaluated from the nonlocal equivalent strain $\bar{\varepsilon}_{\mathrm{eq}}$, but the strain $\boldsymbol{\varepsilon}$ that appears in (1) explicitly is kept local. In the elastic range, the damage variable remains equal to zero, and the stress-strain relation is local.

Damage model with combination of local and nonlocal equivalent strains

As a new development, we propose a new type of nonlocal damage model, with damage driven by a combination of the local and nonlocal values of equivalent strain. In a first attempt, we consider a weighted arithmetic average of $\varepsilon_{\mathrm{eq}}$ and $\bar{\varepsilon}_{\mathrm{eq}}$. The loading function is written as

$$f(\bar{\varepsilon}_{\mathrm{eq}}, \varepsilon_{\mathrm{eq}}, \kappa) = \beta\bar{\varepsilon}_{\mathrm{eq}} + (1-\beta)\varepsilon_{\mathrm{eq}} - \kappa \tag{11}$$

where β is a dimensionless model parameter. This new formulation is inspired by the so-called over-nonlocal plasticity model [4, 3], which deals with a similar linear combination of the local and nonlocal cumulative plastic strains. It covers, as special cases, the local model (3) with $\beta = 0$ and the standard nonlocal model (10) with $\beta = 1$.

One-dimensional localization analysis

To shed more light on the regularizing effect of the nonlocal term and on the role of parameter β, we will perform a simple one-dimensional analysis of bifurcations from a uniform state under uniaxial tension. This is just an academic example, but it is instructive because it can be solved semi-analytically and the results indicate which parameters influence the size of the localized zone.

We consider a perfectly homogeneous bar with constant cross section, subjected to increasing total elongation. No body or inertia forces are taken into account, and so the stress state is uniform. The problem always admits a solution with uniform strain. The main questions to be addressed here are (i) under which conditions the strain distribution can become nonuniform and (ii) how such nonuniform solutions look like. We analyze the state at incipient loss of strain uniformity, when the current strain is still uniform but the strain rate is not.

The rate form of the stress-strain law (1) reduced to one spatial dimension and combined with the rate form of the damage law (2) reads

$$\dot{\sigma} = (1-\omega)E\dot{\varepsilon} - E\varepsilon\dot{\omega} = E_{\rm u}\dot{\varepsilon} - E\varepsilon g'\dot{\kappa} \tag{12}$$

where $E_{\rm u} = (1-\omega)E$ is the unloading (secant) modulus and $g' = {\rm d}g/{\rm d}\kappa$ is the derivative of the damage function g.

Consider first the local damage model. Up to the current state, the material has experienced no unloading and there is no difference between the damage-driving variable κ and the longitudinal strain ε. But the rate $\dot{\kappa}$ is equal to the strain rate $\dot{\varepsilon}$ only if the strain keeps growing, otherwise we have $\dot{\kappa} = 0$. Both cases are covered by the relation $\dot{\kappa} = \langle\dot{\varepsilon}\rangle$. In the damage zone, characterized by $\dot{\varepsilon} > 0$, expression (12) for the stress rate can be written as $\dot{\sigma} = E_{\rm ed}\dot{\varepsilon}$ where $E_{\rm ed} = E_{\rm u} - E\varepsilon g'$ is the elasto-damage (tangent) modulus. Replacing $-E\varepsilon g'$ by $E_{\rm ed} - E_{\rm u}$, we can rewrite (12) as

$$\dot{\sigma} = E_{\rm u}\dot{\varepsilon} + (E_{\rm ed} - E_{\rm u})\dot{\kappa} \tag{13}$$

Let us now turn attention to the nonlocal model. For the formulation presented in the preceding section, damage is driven by the linear combination of local and nonlocal equivalent strains, and the rate of the internal variable κ at the onset of localization can be expressed as $\dot{\kappa} = \langle\beta\dot{\bar{\varepsilon}} + (1-\beta)\dot{\varepsilon}\rangle$. Relation (13) written for the entire bar provides an equation that governs the distribution of the strain rate. This equation, in general written as

$$E_{\rm u}\dot{\varepsilon}(x) + (E_{\rm ed} - E_{\rm u})\langle\beta\dot{\bar{\varepsilon}}(x) + (1-\beta)\dot{\varepsilon}(x)\rangle = \dot{\sigma} \tag{14}$$

has an integral character. Note that the current values of the moduli $E_{\rm u}$ and $E_{\rm ed}$ are taken as constants because at the onset of localization the current state is still uniform, and that the stress rate $\dot{\sigma}$ is also independent of the spatial coordinate—its uniformity follows from the equilibrium equation. Depending on the sign of the stress rate and of the tangent modulus, equation (14) has uniform solutions $\dot{\varepsilon}(x) = \dot{\sigma}/E_{\rm ed} > 0$ or $\dot{\varepsilon}(x) = \dot{\sigma}/E_{\rm u} < 0$ that correspond to uniform damage growth or to uniform elastic unloading, respectively. We will now look for possible nonuniform solutions.

For a given ratio $E_{\rm ed}/E_{\rm u}$, the exact width and shape of the strain rate profile can be found numerically. This will later be done by the finite element method, but it is instructive to start from a theoretical analysis and try to solve equation (14) directly, with $\dot{\sigma}$ considered as a given (negative) constant. The most interesting solution is that for which the growth of damage localizes into an interval $I_{\rm d}$ of finite length $L_{\rm d}$, surrounded by elastically unloading material with constant damage. The origin of the coordinate system will be placed in the center of the localized damage zone, which means that $I_{\rm d}$ will be considered as the interval $(-L_{\rm d}/2, L_{\rm d}/2)$.

For simplicity, we assume that the total bar length L is sufficiently large, so that $I_{\rm d}$ is contained in the interval $(-L/2, L/2)$ representing the entire bar. Outside $I_{\rm d}$, the rate of the loading function is nonpositive and (14) reduces to $E_{\rm u}\dot{\varepsilon}(x) = \dot{\sigma}$, from which $\dot{\varepsilon}(x) = \dot{\sigma}/E_{\rm u}$ for all $x \notin I_{\rm d}$. It is convenient to introduce a new unknown function $\dot{e}(x) = \dot{\varepsilon}(x) - \dot{\sigma}/E_{\rm u}$ (which is nonvanishing only in $I_{\rm d}$) and rewrite (14) as

$$\left(\beta + \frac{E_{\rm ed}}{E_{\rm u} - E_{\rm ed}}\right)\dot{e}(x) - \beta\dot{\bar{e}}(x) = \frac{\dot{\sigma}}{E_{\rm u}} \qquad \text{for } x \in I_{\rm d} \tag{15}$$

Physically, $\dot{e}$ can be considered as the damage strain rate or inelastic strain rate, because the rate form of the stress-strain law can be presented as $\dot{\sigma} = E_{\rm u}(\dot{\varepsilon} - \dot{e})$.

Since $\dot{e}(x)$ vanishes outside the interval $I_d = (-L_d/2, L_d/2)$, the nonlocal term can be expressed by an integral over I_d and (15) can be presented as

$$\left(\beta + \frac{E_{ed}}{E_u - E_{ed}}\right)\dot{e}(x) - \beta\int_{-L_d/2}^{L_d/2}\alpha(x,\xi)\dot{e}(\xi)\mathrm{d}\xi = \frac{\dot{\sigma}}{E_u} \qquad \text{for } -L_d/2 \le x \le L_d/2 \tag{16}$$

This is a Fredholm integral equation of the second kind with a constant right-hand side. The nonstandard feature is that the interval I_d on which (16) is to be solved is not known in advance—it must be found from the loading-unloading conditions, which require that $I_d = \{x \,|\, \beta\dot{\bar{\varepsilon}}(x) + (1-\beta)\dot{\varepsilon}(x) > 0\}$, i.e., in terms of the new unknown function, that $I_d = \{x \,|\, \beta\dot{\bar{e}}(x) + (1-\beta)\dot{e}(x) > -\dot{\sigma}/E_u\}$.

A problem having the same mathematical structure was analyzed in detail in [2]. The general results derived in that paper can be applied to the present case. It was shown that if the coefficient multiplying the local term is positive, the local rate $\dot{e}(x)$ remains continuous (not only inside the interval I_d but everywhere), which means that its value at the boundary of I_d is zero. This is the condition from which the actual size of the damage zone L_d can be determined. Furthermore, a localized solution (i.e. a solution for which I_d does not extend over the entire bar) exists only if (i) the right-hand side of (16) is nonpositive and (ii) the sum of the coefficients multiplying the local term and the nonlocal one is also nonpositive. The sign of the right-hand side depends on the stress rate, because E_u is always positive. Therefore, localization may occur only at non-increasing stress. The second condition leads to $E_{ed}/(E_u - E_{ed}) \le 0$, which is equivalent to $E_{ed} \le 0$. Thus, for the present one-dimensional model, the onset of localization coincides with the onset of softening. The localized solution is continuous if

$$\beta + \frac{E_{ed}}{E_u - E_{ed}} > 0 \tag{17}$$

Condition (17) can be rewritten in terms of the ratio between the tangent and unloading moduli. For $\beta < 1$ we obtain

$$\frac{E_{ed}}{E_u} > -\frac{\beta}{1-\beta} \tag{18}$$

while for $\beta \ge 1$ the condition is always satisfied. So we conclude that the standard nonlocal model with $\beta = 1$ and the "over-nonlocal" model with $\beta > 1$ lead to solutions with the strain localized but still continuous, while the "under-nonlocal" model with $\beta < 1$ can lead to localized solutions with discontinuities in strain, provided that the ratio between the tangent and unloading moduli drops below a certain negative critical value. The critical ratio E_{ed}/E_u depends on β and for β approaching 1 it tends to minus infinity while for β approaching zero it tends to zero from below. Models with small β are "lightly regularized" and preserve continuity of strain only for very mild softening. Models with larger β are "moderately regularized" and develop a strain discontinuity only for dramatic softening. Models with $\beta \ge 1$ are "fully regularized" and the strain profile remains continuous even for dramatic softening.

Evolution of the strain profile

The actual value of the ratio between the tangent and unloading moduli depends on the shape of the stress-strain diagram and on the strain level. In the elastic range, this ratio is 1, and it becomes negative after the peak of the stress-strain diagram. In the pre-peak range, no bifurcation is possible and the solution remains uniform. For stress-strain curves with a sharp peak, e.g. those described by the damage law (4), the slope at peak is right away negative and a bifurcation occurs. The localized

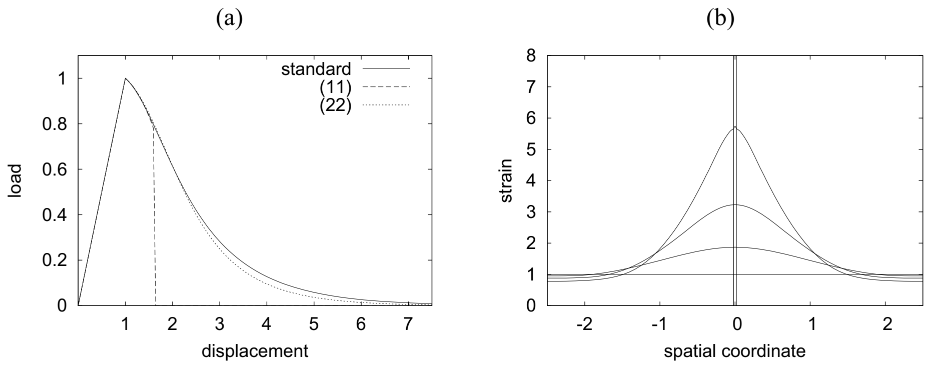

Figure 1: (a) Load-displacement diagrams for three different nonlocal models, (b) evolution of the strain profile for model with loading function (11).

solution remains continuous if the softening is not too dramatic. In terms of the damage function g and its derivative g', the ratio between the tangent and unloading moduli can be expressed as

$$\frac{E_{\rm ed}}{E_{\rm u}} = \frac{(1-g)E - E\varepsilon g'}{(1-g)E} = 1 - \frac{\varepsilon g'}{1-g} \tag{19}$$

For the damage law (4) we get

$$g'(\varepsilon) = \left(\frac{\varepsilon_0}{\varepsilon^2} + \frac{\varepsilon_0}{\varepsilon}\frac{1}{\varepsilon_{\rm f}-\varepsilon_0}\right)\exp\left(-\frac{\langle\varepsilon-\varepsilon_0\rangle}{\varepsilon_{\rm f}-\varepsilon_0}\right) \tag{20}$$

$$\frac{E_{\rm ed}}{E_{\rm u}} = 1 - \frac{\varepsilon g'(\varepsilon)}{1-g(\varepsilon)} = 1 - \left(1 + \frac{\varepsilon}{\varepsilon_{\rm f}-\varepsilon_0}\right) = -\frac{\varepsilon}{\varepsilon_{\rm f}-\varepsilon_0} \tag{21}$$

The above relations are valid for $\varepsilon \geq \varepsilon_0$. Right at peak, we have $\varepsilon = \varepsilon_0$ and $E_{\rm ed}/E_{\rm u} = -\varepsilon_0/(\varepsilon_{\rm f}-\varepsilon_0)$. Condition (18) is satisfied if $\varepsilon_{\rm f} > \varepsilon_0/\beta$. So the initial localized strain distribution remains continuous if the ratio $\varepsilon_{\rm f}/\varepsilon_0$ that controls the relative steepness of the softening branch of the stress-strain diagram exceeds the inverse value of parameter β. But it is interesting to note that, with increasing ε, the ratio $E_{\rm ed}/E_{\rm u}$ evaluated according to (21) tends to minus infinity. Consequently, condition (18) is always violated if the test proceeds far enough.

Strictly speaking, condition (18) refers only to a bifurcation from a uniform state, so it would be applicable only if the strain distribution was artificially held uniform up to the current state. Then, for strains exceeding $(\varepsilon_{\rm f}-\varepsilon_0)\beta/(1-\beta)$, a discontinuous bifurcation could take place. Intuitively it can be expected that if the strain is not kept uniform, the localized profile with a continuous strain distribution will evolve towards a discontinuity. However, the actual model response needs to be verified by numerical simulations.

All simulations are performed using a one-dimensional model of a bar of length $L = 8$ discretized by finite elements of minimum size 0.0125. Localization is triggered by reducing the cross-sectional area of one element by 0.1%. The bar is loaded by monotonically increasing applied displacement at its right end while the left end is kept fixed. All quantities plotted in the graphs are normalized: the displacement and force are divided by their peak values, the strains are divided by the strain at peak stress, etc. An exponential damage law (4) with $\varepsilon_f = 10\,\varepsilon_0$ is adopted. The nonlocal interaction radius is initially set to $R = 1$.

The solid curve in Fig. 1a represents the load-displacement diagram for the standard nonlocal model with $\beta = 1$ and $R = 1$. The dashed curve corresponds to the new model with $\beta = 0.5$ and

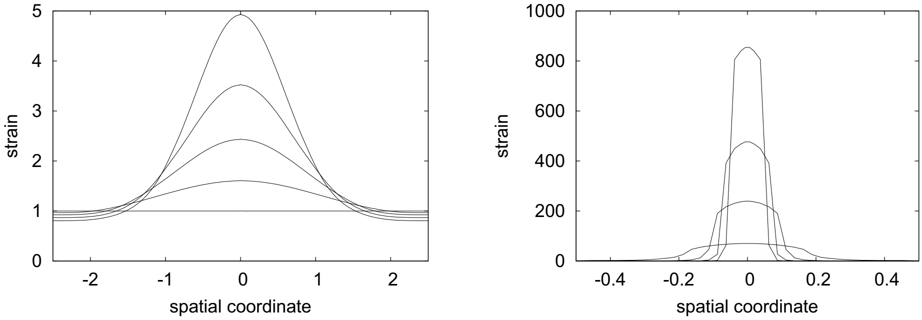

Figure 2: Nonlocal model with loading function (22) and parameter $\beta = 0.5$: evolution of strain profile at (a) early and (b) late stages of softening.

$R = 1.5$. The interaction radius is increased to compensate for the effect of β on the size of the damage zone. With parameters $\beta = 0.5$ and $R = 1.5$, the initial part of the softening curve remains almost the same as for the standard nonlocal model. However, when the applied displacement exceeds approximately 1.6 times the displacement at peak, the stress transmitted by the bar suddenly drops to zero and the bar fails. This sudden failure is related to an abrupt localization of strain increments into one single element; see Fig. 1b. So the initial version of the new model suffers by a serious deficiency.

The pathological behavior is eliminated if the damage-driving quantity is redefined as the harmonic average of the local and nonlocal equivalent strain instead of the arithmetic average. Equation (11) is reformulated as

$$f(\bar{\varepsilon}_{\text{eq}}, \varepsilon_{\text{eq}}, \kappa) = \left(\beta\bar{\varepsilon}_{\text{eq}}^{-1} + (1-\beta)\varepsilon_{\text{eq}}^{-1}\right)^{-1} - \kappa \tag{22}$$

After this modification, the load-displacement diagram becomes continuous and turns out to be very close to the original one; see the dotted curve in Fig. 1a. The strain profile is continuous during the initial stages of localization (Fig. 2a) but later the strain increments concentrate in a small part of the damage zone between two weak discontinuities (Fig. 2b). Note the different scales on the axes in Figs. 2a and 2b. At late stages of softening, the active part of the damage zone is very small and the solution tends to a localized crack. The evolution of the damage-driving variable κ is documented in Fig. 3a, and the final damage distribution is plotted in Fig. 3b. The dotted curve in Fig. 3b corresponds to the standard nonlocal model, for which the damage variable always tends to 1 in an interval of size $2R$. This deficiency is alleviated by the newly proposed formulation.

Acknowledgment

Financial support of the Czech Science Foundation under Project GAČR 106/08/1508 is gratefully acknowledged.

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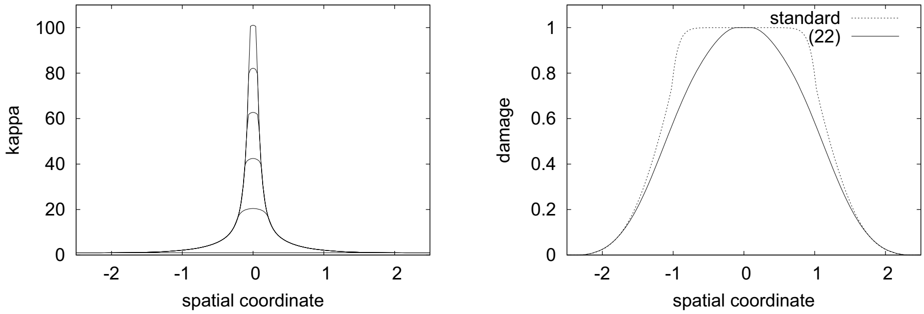

Figure 3: Nonlocal model with loading function (22) and parameter $\beta = 0.5$: (a) evolution of the damage-driving internal variable κ, (b) final distribution of damage (compared with the standard nonlocal model.

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