Crack propagation in anisotropic inhomogeneous 2-D-structures

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Abstract. In this contribution we present ideas how fatigue crack growth in anisotropic composite materials can be predicted using the Griffith' energy criteria for plane problems. We assume, that the specimen under consideration is composed of two anisotropic materials with different elastic properties. If the crack path is increased by a small crack shoot under an external load, recent mathematical investigations show, that the change of the potential energy can be calculated by stress intensity factors and some integral characteristics, if the crack and the boundary layer are not in contact.
To predict the crack propagation process in a vicinity of the interface, calculations of stress intensity factors are needed and therefor, one has to know the asymptotic expansion of the displacement field near the crack tip, if the crack reaches the boundary layer between the two materials. An idea how this can be done will extensively be discussed.

Introduction

The growing use of non-homogeneous anisotropic materials, e.g. composites, laminates or functionally graded materials, in order to fulfill the requirements of modern engineering has given an impulse to the study of fracture mechanisms in such structures. For means of a reliable fracture mechanical assessment the simulation of crack propagation processes is necessary. Especially for composites, the following behavior can be observed: When a crack tip touches an interface, the crack can further propagate through the adjacent material, propagate along the interface or stop. From a physical point of view the energy principle, already formulated by Griffith in 1921, can be applied in anisotropic and inhomogeneous materials to compute the crack path and the behavior near the interface:

A crack is growing in such a way that the total energy always is minimal.

The total energy Π is composed from the surface energy S and the potential energy U, the latter is the difference of the elastic energy and the work performed by external forces.

For homogeneous solids, recent mathematical investigations showed the following: Suppose the crack path is increased by a (small) crack shoot of length h, then the change of the potential energy can be calculated asymptotically to [1]

$$\Delta U = -\frac{1}{2}\mathbf{K}^{\top}\mathbb{M}(h)\mathbf{K} + \mathcal{O}(h^{N+1/2}), \qquad h \to 0. \tag{1}$$

Thereby $\mathbf{K}$ denotes the vector of stress intensity factors, N is the number of terms used in the asymptotic decomposition of the displacement field near the crack tip and $\mathbb{M}(h)$ is a symmetric matrix, the so called energy release matrix (ERM). The ERM contains certain integral characteristics depending on the geometry of the specimen and the crack shoot as well as the elastic properties of the material. All entries of ERM can be calculated numerically up to sufficient precision. Using the asymptotic energy release rate the kink angle of a crack can be determined in arbitrary plane anisotropies and the crack path can be approximated by polygons piecewise.

If the specimen under consideration is composed of two materials with different material properties, the same formula for the change of potential energy holds, if the crack and the boundary layer are not in contact. In a composite material the correct calculation of stress intensity factors and the ERM is a bit more complicated as in the homogeneous case, even if the crack tip has large distance to the interface. Nevertheless, formula (1) can be used to compute the crack path away from the contact surface.

As mentioned, the behavior of the crack near the boundary layer is of importance. Recent results, e.g. [2], show that stress intensity factors can go to zero or grow up to infinity if the crack tip is in a vicinity of the contact surface, depending on the material properties. Numerical computations can only approximate this and are very extensive. To determine the behavior of the developing crack path, one has to know the asymptotic expansion of the displacement field, if the crack tip is on the interface. But there, the singular behavior of the displacement field changes and is not known yet for arbitrary anisotropies.

In the following, we discuss ideas to detect the influence of the inhomogeneity on the developing crack path. Introducing a small parameter in the material properties, which can be interpreted as a measure for the level of inhomogeneity, we single out formulae for the correct numerical calculation of stress intensity factors in a composite first. The influence of the inhomogeneity will be shown. Carrying forward this idea of an inhomogeneity-parameter, we present how the asymptotic behavior of the displacement field can be calculated, if the crack tip is on the contact surface. Using methods of asymptotic analysis, we finally derive formulae for the behavior of stress intensity factors near the interface, only by knowing the elastic moduli of the two materials.

Formulation of the problem

Let $\Omega \subset \mathbb{R}^2$ be a bounded, polygonal domain with outer boundary Γ, composed of two linear elastic homogeneous solids Ω^0 and Ω^δ with different material properties. For simplicity, we assume that the boundary layer is at $x_1 = 0$. Then the Hooke-tensor of this composite solid is piecewise smooth with

$$A(x) := \begin{cases} A^0, & x_1 < 0, \\ A^\delta, & x_1 \geq 0, \end{cases}$$

where A^0 and A^δ are constant, containing the elastic moduli of Ω^0 and Ω^δ. To investigate the influence of the inhomogeneity on the crack propagation process, we introduce a small parameter δ and assume

$$A^\delta = A^0 + \delta A^1, \qquad |\delta| < 1.$$

Here, A^k are symmetric positive definite matrices containing the elasticity constants a^k_{ij}, $k = 0, 1$, $i, j = 1, 2, 3$. The matrix A^δ is assumed to be positive definite also. For isotropic materials there holds

$$a^k_{11} = a^k_{22} = \lambda^k + 2\mu^k, \qquad a^k_{21} = \lambda^k, \qquad a^k_{33} = \mu^k, \qquad a^k_{31} = a^k_{32} = 0, \qquad k = 0, 1,$$

and λ^k, μ^k are the Lamé-constants of Ω^k. For small parameter δ, the inhomogeneity can be understand as a perturbation of the material and δ can be interpreted as a measure for this perturbation. This is a first step to model functionally graded materials or other more complex structures.

Our main interest is the behavior of a rectilinear edge cut

$$\Xi_h := \{x \in \overline{\Omega} : x_1 \le -h, x_2 = 0\}, \quad h \ge 0, \qquad \Omega_h := \Omega \setminus \Xi_h,$$

where the crack tip is close to the boundary layer and h assumed as small. We consider the problem of 2-dimensional elasticity theory:

$$\begin{aligned} -\nabla \cdot \sigma(u_h; x) &= 0, \quad x \in \Omega_h, \\ \sigma(u_h; x) \cdot n &= 0, \quad x \in \Xi_h^+ \cup \Xi_h^-, \\ \sigma(u_h; x) \cdot n &= p(x), \quad x \in \Gamma. \end{aligned}$$

Here, u_h is the displacement field, p is the vector of surface load, n the outward normal vector and with $\Xi_h^\pm$ we denote the crack faces, assumed to be stress-free. On the contact surface

$$\Upsilon := \{x \in \overline{\Omega} : x_1 = 0\}$$

the displacement field u_h fulfills the jump conditions

$$u_h(0_-, x_2) - u_h(0_+, x_2) = 0,$$

$$\sigma_{1k}(u_h; 0_-, x_2) - \sigma_{1k}(u_h; 0_+, x_2) = 0, \qquad k = 1, 2.$$

Moreover, we assume, that the load is self-balanced and fulfills the compatibility conditions

$$\int_\Gamma p(x) \cdot v(x)\, ds = 0$$

for all rigid motions $v = a + b\binom{x_2}{-x_1}, a \in \mathbb{R}^2, b \in \mathbb{R}$. For the strain-tensor with components

$$\varepsilon_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \qquad i, j = 1, 2$$

we use the vector notation

$$\varepsilon(u; x) = \Big(\varepsilon_{11}(u; x), \varepsilon_{22}(u; x), \sqrt{2}\varepsilon_{12}(u; x)\Big)^\top.$$

The stress- and strain-tensors are connected by Hooke's Law:

$$\sigma(u; x) \;=\; \Big(\sigma_{11}(u; x), \sigma_{22}(u; x), \sqrt{2}\sigma_{12}(u; x)\Big)^\top = A(x) \cdot \varepsilon(u; x).$$

For abbreviation, we set

$$\sigma^\delta(u; x) := A^\delta \cdot \varepsilon(u; x), \qquad |\delta| < 1,$$

$$\sigma^1(u; x) := A^1 \cdot \varepsilon(u; x).$$

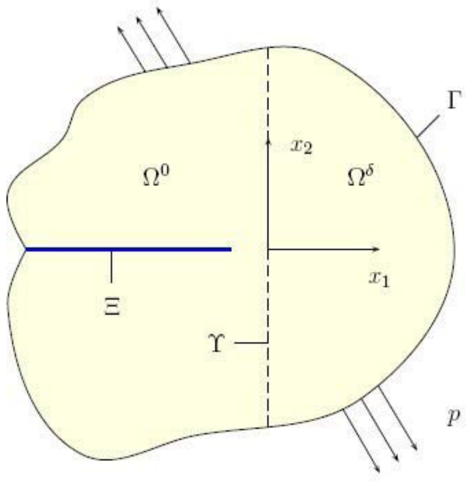

Figure 1: The inhomogeneous solid Ω_h

Asymptotics of the displacement field

If the crack is not in contact with the boundary layer, the displacement field u_h has an asymptotic expansion of the following well know type near the crack tip [3]:

$$u_h(\hat{x}) = r^{1/2}\left(K_I(h)\Phi^1(\varphi) + K_{II}(h)\Phi^2(\varphi)\right) + \dots, \qquad r \to 0, \tag{2}$$

where $\hat{x} = (x_1 + h, x_2)^\top$ are local coordinates at the crack tip and (r, φ) associated polar coordinates. $K_I(h)$ and $K_{II}(h)$ are the stress intensity factors, depending on h and, of course, on the inhomogeneity . Φ^j are smooth functions, depending only on the angle φ. The functions

$$X^j(\hat{x}) = r^{1/2}\Phi^j(\varphi), \qquad j = 1, 2,$$

are solutions of the homogeneous elasticity problem in the whole plane with a semi-infinite cut

$$-\nabla \cdot \sigma^0(X^j;\hat{x}) = 0, \qquad \hat{x} \in \mathbb{R}^2 \setminus \Lambda, \qquad \Lambda := \{\hat{x} : \hat{x}_1 \le 0, \hat{x}_2 = 0\},$$

$$\sigma^0_{12}(X^j;\hat{x}_1, 0) = 0, \qquad \sigma^0_{22}(X^j;\hat{x}_1, 0) = 0, \qquad \hat{x}_1 < 0.$$

This functions are called eigenfunctions of the elasticity operator and depend only on the material properties of Ω^0. Moreover, there exist singular eigenfunctions of the elasticity operator of the form

$$Y^j(\hat{x}) = r^{-1/2}\Psi^j(\varphi), \qquad j = 1, 2,$$

fulfilling the normalization condition

$$\int_\gamma \left(\sigma^0(X^i;x)\cdot n(x)\right)\cdot Y^j(x) - \left(\sigma^0(Y^j;x)\cdot n(x)\right)\cdot X^i(x)\, ds = \delta_{i,j}, \qquad i,j = 1,2,$$

where $\delta_{i,j}$ is the Kronecker symbol and γ is a smooth path around the crack tip, starting in Ξ_h^-, ending in Ξ_h^+ and lying in Ω^0. for more details see e.g. [4],[5]. If the specimen under consideration is homogeneous ($\delta = 0$, for example), stress intensity factors can be calculated with the help of these singular functions [6],[7]:

$$K_j(h) = \int_\gamma p(x) \cdot Y^j(x_1 + h, x_2)\, ds - \int_\gamma u_h(x) \cdot \left(\sigma^0(Y^j; x_1 + h, x_2) \cdot n(x)\right)\, ds,$$

$j = 1, 2$, $h > 0$. Using Green's formula, this relation can be extended to the inhomogeneous case:

$$K_j(h) = \int_\Gamma p(x) \cdot Y^j(x_1 + h, x_2)\, ds - \int_\Gamma u_h(x) \cdot \left(\sigma^0(Y^j; x_1 + h, x_2) \cdot n(x)\right)\, ds \quad (3)$$

$$-\delta \left(\int_{\partial\Omega^\delta} \left(\sigma^1(u_h; x) \cdot n(x)\right) \cdot Y^j(x; x_1 + h, x_2)\, ds \right), \qquad j = 1, 2. \quad (4)$$

This relation exactly shows the influence of the parameter δ on the stress intensity factors. We emphasize, that Y^j only depend on the material properties of Ω^0. Eigenfunctions are known exactly, if the material is isotropic. For some special classes of anisotropic materials, explicit formulae for X^j and Y^j are given in [8] and for all other anisotropies, eigenfunctions can be computed numerically up to arbitrary precision. The stress intensity factors can be calculated very precisely using formula (3). Only the displacement field has to be computed numerically, for example by the Finite-Element-Method. For more details we refer to [9].

Asymptotics of the displacement field on the boundary layer

If the crack reaches the interface line ($h = 0$), the asymptotic expansion (2) of the displacement changes. The expansion still has the structure [4]

$$u(x; \delta) = K_I^\delta X^1(x; \delta) + K_{II}^\delta X^2(x; \delta) + \ldots, \qquad x \to 0.$$

The functions $X^j(\cdot; \delta)$ are solutions of the homogeneous elasticity problem in the whole composite plane:

$$-\nabla \cdot \sigma(X^j(\cdot; \delta); x) = 0, \qquad x \in \mathbb{R}^2 \setminus \Lambda,$$

$$\sigma_{12}(X^j(\cdot; \delta); x_1, 0) = 0, \qquad \sigma_{22}(X^j(\cdot; \delta); x_1, 0) = 0, \qquad x_1 < 0.$$

They fulfill the jump conditions

$$X^j(0_-, x_2; \delta) - X^j(0_+, x_2; \delta) = 0, \qquad x_2 \in \mathbb{R},$$

$$\sigma_{1k}(X^j(\cdot; \delta); 0_-, x_2) - \sigma_{1k}(X^j(\cdot; \delta); 0_+, x_2) = 0, \qquad k = 1, 2.$$

For small parameter δ, these functions will not differ a lot from X^j of the homogeneous case. To calculate $X^j(\cdot; \delta)$, we use the ansatz

$$X^j(x; \delta) = r^{1/2 + \alpha_j \delta + \ldots} \left(\Phi^{j0}(\varphi) + \delta \Phi^{j1}(\varphi) + \ldots\right), \qquad j = 1, 2, \quad (5)$$

where α_j are constants. Expanding this expression in a power series, a short calculation shows

$$X^j(x; \delta) = r^{1/2} \Phi^{j0}(\varphi) + \delta \left(\alpha_j \ln(r) r^{1/2} \Phi^{j0}(\varphi) + r^{1/2} \Phi^{j1}(\varphi)\right) + \mathcal{O}(\delta^2)$$

and we are only interested in the first terms of this expansion. Taking into account this structure, we look for the first angular parts Φ^{j0} as a combination of the angular parts of the homogeneous case:

$$r^{1/2}\Phi^{j0}(\varphi) = r^{1/2}\sum_{p=1}^{2} B_p^j \Phi^p(\varphi) = \sum_{p=1}^{2} B_p^j X^p(x).$$

Hence, in a first step, it remains to compute the matrix $B = (B_p^j)_{jp}$, the numbers α_j and the perturbation angular parts Φ^{j1}.

Further computations show, that the ansatz (5) can only work, if for α_j and B^j the relation

$$\alpha_j B^j = S \cdot B^j$$

holds, where S is the matrix with components

$$S_{kp} = -\sum_{j=1}^{2}(M^{-1})_{kj}^{\top} \int_{\gamma} \varepsilon(X^p; x) \cdot A^1 \cdot \varepsilon(\partial_1 X^j; x)\, ds, \qquad k,p = 1,2,$$

where

$$\gamma := \{x : |x| = 1, x_1 \geq 0\}$$

is the half-arc. Matrix M is composed of the constants from the relation [1],[10]

$$-\partial_{x_1} X^j(x) = \sum_{k=1}^{2} M_{jk} Y^k(x), \qquad \text{or} \qquad Y^k(x) = -\sum_{j=1}^{2}(M^{-1})_{kj}^{\top}\partial_{x_1} X^j(x).$$

We emphasize, that the functions X^j and Y^k depend only on the material properties of A^0.

Explicit formulae and behavior of the stress intensity factors

If Ω^0 is an isotropic solid with

$$A^0 = \begin{pmatrix} \lambda + 2\mu & \lambda & 0 \\ \lambda & \lambda + 2\mu & 0 \\ 0 & 0 & 2\mu \end{pmatrix},$$

then Φ^j are known exactly,

$$\Phi^1(\varphi) = \frac{1}{4\sqrt{2\pi}\mu(\lambda+\mu)} \begin{pmatrix} -(\lambda+\mu)\cos(3\varphi/2) + (\lambda+5\mu)\cos(\varphi/2) \\ -(\lambda+\mu)\sin(3\varphi/2) + (3\lambda+7\mu)\sin(\varphi/2) \end{pmatrix},$$

$$\Phi^2(\varphi) = \frac{1}{4\sqrt{2\pi}\mu(\lambda+\mu)} \begin{pmatrix} (\lambda+\mu)\sin(3\varphi/2) + (5\lambda+9\mu)\sin(\varphi/2) \\ -(\lambda+\mu)\cos(3\varphi/2) + (\lambda-3\mu)\cos(\varphi/2) \end{pmatrix},$$

and there holds

$$M_{11} = M_{22} = \frac{\lambda + 2\mu}{2\mu(\lambda+\mu)}, \qquad M_{12} = M_{21} = 0.$$

The matrix S can be computed directly and the computation of α_j and B^j is only to calculate eigenvalues and eigenvectors of a 2×2-matrix. If A^1 is the Hooke-tensor of an orthotropic material,

$$A^1 = \begin{pmatrix} a_{11} & a_{21} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 2a_{33} \end{pmatrix},$$

we find

$$\alpha_1 = \frac{\lambda^2(a_{11} - 2a_{21} + a_{22} + 4a_{33}) - 2\lambda\mu(a_{11} + 2a_{21} - 3a_{22} - 4a_{33})}{32\pi\mu(\lambda+\mu)(\lambda+2\mu)} + \frac{\mu^2(a_{11} + 6a_{21} + 9a_{22} + 4a_{33})}{32\pi\mu(\lambda+\mu)(\lambda+2\mu)}$$

$$\alpha_2 = \frac{\lambda^2(a_{11} - 2a_{21} + a_{22} + 4a_{33}) - 2\lambda\mu(-3a_{11} + 2a_{21} + a_{22} - 4a_{33})}{32\pi\mu(\lambda+\mu)(\lambda+2\mu)} + \frac{\mu^2(9a_{11} + 6a_{21} + a_{22} + 4a_{33})}{32\pi\mu(\lambda+\mu)(\lambda+2\mu)}$$

and

$$B^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad B^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

This idea can be extended to more complicated geometries of contact surfaces.

Finally, the idea of introducing the parameter δ and ansatz (5) gives a way to present the behavior of the stress intensity factors, if the crack tip is near the boundary layer. Using methods of asymptotic analysis [4],[5], the following formula for the stress intensity factors holds:

$$K_I(h) = \mathcal{O}(h^{\delta\alpha_1+\cdots}), \qquad K_{II}(h) = \mathcal{O}(h^{\delta\alpha_2+\cdots}), \qquad h \to 0.$$

For small parameter δ this relation, very easy to handle, reflects the asymptotic behavior of stress intensity factors and of the crack path near the interface. It extends the results from [2]. Depending on α_j, the stress intensity factors will go to zero or grow up to infinity or, if α_j vanishes, they will show a steady behavior. This is a key to decide, wether the crack can propagate through or along the interface. But of course, for this decision one has to take into account properties of the contact zone itself.

Conclusions

In this paper, an idea for the computation of quasi-static crack growth in anisotropic composite materials is discussed. Introducing a parameter in the material properties in order to measure the level of inhomogeneity, formulae for the numerical computation of stress intensity factors are presented. Proceeding with this idea, the asymptotic behavior of the displacement field can be calculated, if the crack tip reaches the contact surface. Finally, the asymptotic behavior of the stress intensity factors is shown. In practical applications this results can be used to setting up specialised demands on structural components.

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