Crack opening and crack flank displacements in the prefracture zone when branching and kinking

V.D. Kurguzov [a] and V.M. Kornev [b]

Lavrentyev Institute of Hydrodynamics SB RAS, Novosibirsk, 630090, Russia

[a]kurguzov@hydro.nsc.ru, [b]kornev@hydro.nsc.ru

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Abstract. Solids with regular structure are under consideration. Solution of nonlinear problem of fracture mechanics is supposed to be obtained in two steps when branching or kinking is possible for mode I crack. First, branch or kink angles are searched, and then, critical fracture parameters for chosen direction are determined. At these steps, the necessary and sufficient fracture criteria of the Neuber-Novozhilov type, respectively, are used. When searching critical fracture parameters (prefracture zone lengths and loading), the modification of the classical Leonov-Panasyuk-Dugdale model is used when prefracture zones occupy rectangles located along crack branches. When deriving simple expressions for critical fracture parameters, stress intensity factors (SIFs) for a crack with the infinitesimal branch are used, and SIFs for a crack with a branch are used when normal and shear stresses modeling the plasticity zone are given at the branch. To construct SIFs for a crack with branches, the problem of uniaxial tension of a plate with crack having bisymmetric branching has been solved using the method of finite elements. SIFs by the modes I and II depending on the branch angle have been obtained.

Introduction

Problems of steady growth of sharp cracks and their branching when a solid with the straight sharp crack is under some loading are of a certain interest. In the vicinity of the tip of the sharp mixed mode crack, a stress field occurs. Under certain conditions, bluntness of sharp cracks can take place because of shear stresses or deformations. In work [1], possibility for multiple crack branching is described that is associated with the multiplicity of eigenvalues when buckling the system. We should emphasize in this work that loading corresponds to I mode fracture. Relations describing the branching angle of crack path have been obtained when curves of the theoretical Coulomb-Mohr type strength are known. The crack advances along the following directions: 1) transversely to the direction of maximum tension in the absence of shear stresses in the vicinity of the crack tip (Erdogan-Sih hypothesis) when brittle material behavior takes place; 2) along the direction of maximum shear in the absence of normal stresses in the vicinity of the crack tip when ductile material behavior takes place (emission of dislocations occurs); 3) along some direction corresponding to the generalized stress state when quasi-brittle and quasi-ductile material fracture takes place.

Fig. 1 displays a photograph adopted from the work [2]. The photograph demonstrates axisymmetric formation of prefracture zones in aluminium plate with a crack. Such a picture is observed in thin plates under full-scale plastic flow in the vicinity of the crack tips. Subsequent propagation of cracks takes place along the medial line of neck formation zone.

Branching and kinking of sharp crack paths occur when the tip of a sharp plane crack rests against a plane interface of single crystals. This interface (low-angled boundary, for example) in structured material is considered as some thin solid of the regular structure with given properties. If a thin solid has low strength characteristic as compared with perfect single crystals, then the preferential crack growth coincides with single crystal interface [1].

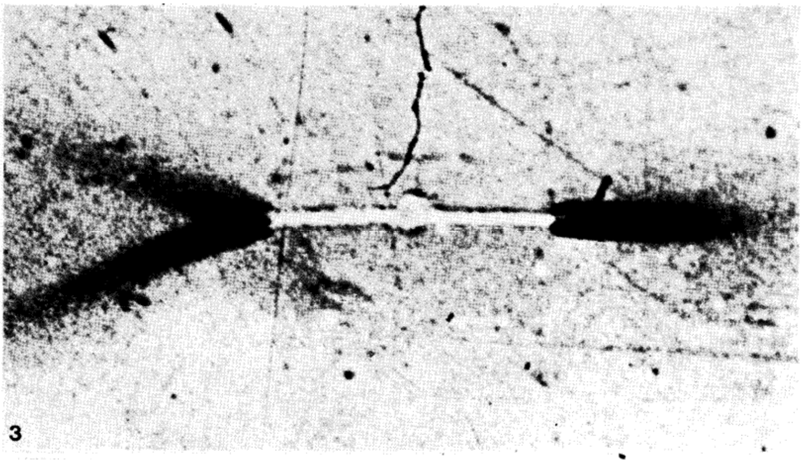

Fig. 1. Asymmetric formation of prefracture zones in aluminium plate with crack.

1. Description of quasi-brittle characteristic of structured material under single loading.

Consider an inner straight sharp crack in structurally inhomogeneous material at the second structural level (granular material). The inner crack in isotropic ductile material is modeled by bilateral cut of $2l$ in length. Let normal stresses σ_∞ and tangential stresses τ_∞ be given at infinity, that is, the crack is deformed in mixed mode. When the crack reaches the interface between two grains, branching or kinking of the crack path is possible as a result of asymmetry of strength material properties with respect to the crack plane. Introduce the polar coordinate system $Or\theta$ with the pole O at the right crack tip, the polar axis being along the crack axis.

By $\pm\theta^*$ denote branch (kink) angles of a crack. For $\theta^* = 0$, the crack extends steadily in straight direction; for $\theta^* \neq 0$, kinking of the crack path takes place; for $\pm\theta^* \neq 0$, the crack branches changing its direction, for $\pm\theta^* = \pi/2$, the crack is blunted when it is opened [1]. By the type of fracture behavior, materials can be divided into brittle ($\theta^* = 0$) and ductile ($\theta^* = \pm\pi/2$), but quasi-brittle ($\theta^* \approx \pm 0, \pm\theta^* \neq 0$) or quasi-ductile ($\theta^* \approx \pm\pi/2, \theta^* < \pi/2$) is also possible.

Under gradual loading with loads σ_∞ and τ_∞ applied to a specimen at infinity, the complex stress state is realized in the vicinity of the crack tip. We will consider proportional loading when $\tau_\infty / \sigma_\infty = c = \text{const}$. The choice of one or another way of branching or kinking of a crack path is conditioned by strength material characteristics. Fig. 2 displays curves of the theoretical strength of the Coulomb-Mohr type for two isotropic materials. The curve 1 describes a behavior of quasi-ductile material, the curve 2 describes a behavior of quasi-brittle material. The proportional way of loading ($\tau/\sigma = c = \text{const}$) is shown with ray 3, the direction of which is defined by the angle φ on the plane $\sigma - \tau$, where σ and τ are normal and shear stresses, respectively. Thus, curves 1 and 2 in the polar coordinate system can be written in the form $f_i = f_i(\varphi)$ ($i = 1, 2$); for isotropic material, we have $f_i(\varphi) = f_i(-\varphi)$ because of satisfying the symmetry conditions. The theoretical (ideal) tension material strengths are denoted σ_{m1} and σ_{m2} for curves 1 and 2, respectively ($f_i(0) = \sigma_{mi}$).The theoretical (ideal) shear material strengths are denoted by τ_{m1} and τ_{m2} for curves 1 and 2, respectively ($f_i(\pi/2) = \tau_{mi}$). By σ^* and τ^* denote critical values of stresses on the given way of deformation. Below values with asterisk denote the critical state.

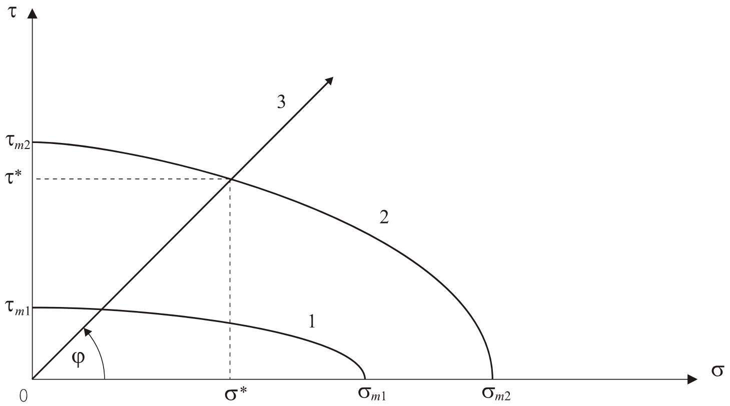

Fig. 2. Theoretical strength curves of Coulomb-Mohr type for two isotropic materials: curve 1 for quasi-brittle material, curve 2 for quasi-ductile material, ray 3 for proportional loading way

Relative estimates of theoretical tension σ_m and shear τ_m strengths in limiting cases are as follows: 1) for materials inclined to cleavage (brittle and quasi-brittle materials), we have $\sigma_m \simeq \tau_m$ (curve 2); 2) for materials weakly resistant to dislocation emission (quasi-ductile material), we have $\sigma_m \gg \tau_m$ (curve1).

For the proportional way of loading, we consider $\tilde{\sigma} - \tilde{\varepsilon}$ fracture assessment curves, where $\tilde{\sigma} = \sqrt{\sigma^2 + \tau^2}$ is the stress intensity, $\tilde{\varepsilon} = \sqrt{\varepsilon^2 + \gamma^2}$ is the deformation intensity. Fracture assessment curves $\tilde{\sigma} - \tilde{\varepsilon}$ can be obtained, for example, in experiments for united tension and torsion of thin-walled pipe specimens [3]. Fig 3 shows the simplest approximation of the real $\tilde{\sigma} - \tilde{\varepsilon}$ diagram of quasi brittle deformation. Here $\tilde{\sigma}^*$ are the critical values of stresses; $\tilde{\varepsilon}_0$ is the limiting deformation in the zone of elastic deformation; $\tilde{\varepsilon}^*$ are deformations corresponding to the beginning of fracture process.

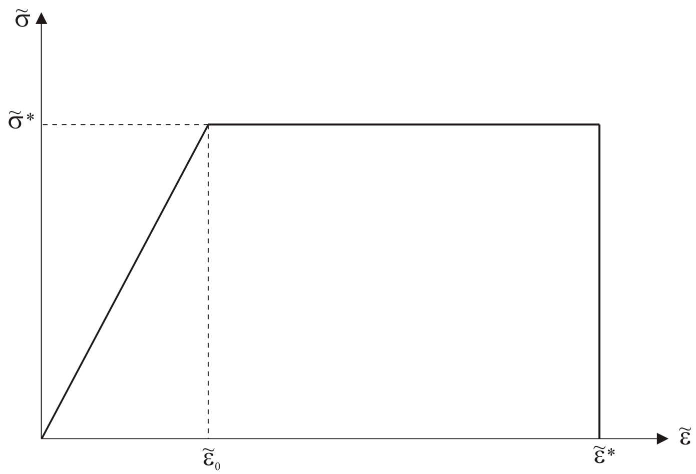

Fig. 3. The simplest approximation of $\tilde{\sigma} - \tilde{\varepsilon}$ diagram of quasi-brittle material deformation.

2. Branch angles of a crack (the necessary fracture criterion)

Suppose the initial macrocrack is sharp and its right tip is on the interface between two grains. The material is supposed to be isotropic.

Consider the strength discrete-integral fracture criterion of the Neuber-Novozhilov type [4, 5] for crack extension in chosen directions $\pm\theta$, which is defined by branch angles

$$\langle \sigma_\theta(\theta) \rangle = \frac{1}{r_0}\int_0^{r_0} \sigma_\theta(r,\theta)dr \le \sigma^*, \quad \langle \tau_{r\theta}(\theta) \rangle = \frac{1}{r_0}\int_0^{r_0} \tau_{r\theta}(r,\theta)dr \le \tau^*. \tag{1}$$

Here $Or\theta$ is a polar coordinate system, the origin of the coordinate system being coincident with the tip of a real crack; $\sigma_\theta(r,\theta)$ and $\tau_{r\theta}(r,\theta)$ are normal and tangential stresses having integrable singularity; $\langle \sigma_\theta(\theta) \rangle$ and $\langle \tau_{r\theta}(\theta) \rangle$ are averaged normal and tangential stresses in the chosen directions $\pm\theta^*$; symbols $\sigma^* = f(\varphi)\cos\varphi$ and $\tau^* = f(\varphi)\sin\varphi$ are used for stresses at critical states (Fig. 2); r_0 is the specific linear size of the material structure (grain diameter). Emphasize that left-hand sides of the first and second relations (1) are functions of the angle θ, and right-hand sides of the same relations are functions of the angle φ. For I mode cracks, the relation between these angles was derived earlier: $\varphi = \theta/2$ [1].

For $\langle \sigma_\theta(\theta) \rangle < \sigma^*$, $\langle \tau_{r\theta}(\theta) \rangle < \tau^*$, the crack does not extend. When the averaged stresses $\langle \sigma_\theta(\theta) \rangle$ and $\langle \tau_{r\theta}(\theta) \rangle$ coincide with stresses of critical states σ^* and τ^*, i.e., $\langle \sigma_\theta(\theta) \rangle = \sigma^*$ and $\langle \tau_{r\theta}(\theta) \rangle = \tau^*$, the criterion (1) is realized in the chosen directions $\pm\theta^*$ and then: 1) the only prefracture zone on the crack continuation is formed if $\theta^* = 0$, 2) two prefracture zones are formed when the inner crack of $2l$ branches, if $\theta^* \ne 0$. The first case corresponds to brittle and quasi-brittle fracture when $\theta^* = 0$. The second case corresponds to quasi-ductile fracture when $\theta^* \simeq \pi/2$.

Using the necessary criterion (1), the length of the sharp inner crack can be written in the form [6]

$$\frac{2l}{r_0} = \frac{nf^2(\varphi)\cos^2\varphi}{\sigma_\infty \cos^3\frac{\theta}{2} - 3\tau_\infty \sin\frac{\theta}{2}\cos^2\frac{\theta}{2}}. \tag{2}$$

Relation (2) contains the function $f(\varphi)$ characterizing material behavior on the plane $\sigma-\tau$ (Fig. 2). The connection between angles φ and θ can be written as [6]

$$\mathrm{tg}\varphi = \frac{K_I \sin\frac{\theta}{2}\cos\frac{\theta}{2} + K_{II}\left(1 - 3\sin\frac{\theta}{2}\right)}{K_I \cos^2\frac{\theta}{2} - 3K_{II}\sin\frac{\theta}{2}\cos\frac{\theta}{2}}, \quad -\pi < \theta < \pi, \tag{3}$$

where $K_I > 0$ and $K_{II} \ne 0$ are SIFs for the sharp inner crack at the generalized stress state.

Using relations (2) and (3), branch angles $\pm\theta^*$ can be determined from the following relation [1]: $l(\theta^*) = \min l(\theta)$. This relation describes just as the simple, so multiple crack branching. For example, for triple branching, we have $l_1(\theta_1^*) = l_2(-\theta_1^*) = l_3(\theta_3^*)$ for $\theta_1^* \ne 0, \theta_3^* = 0$. Such a behavior of the system was revealed in experiments [2]: for the left tip in Fig. 1, we have $\pm\theta^* \ne 0$, and for the right crack tip in Fig. 1, we have $\theta^* = 0$. Implementation of one or another direction of crack extension in particular experiments [2] depends on insufficient perturbations in a material structure in the vicinity of the crack tip.

Thus, a necessary criterion (1) describes the beginning of formation of the only prefracture zone in brittle material or two prefracture zones in quasi-brittle material and allows the angles θ^* of inner crack branching to be determined. The necessary criterion (1) can not describe lengths of prefracture zones.

For the necessary criterion, corresponding averaged stresses are less than theoretical rupture and shear strengths. When the necessary criterion is realized, the material structure nearest to crack tip is in the critical state. However, when critical loading of a structure nearest to the tip is exceeded, the additional extra loading of solid with a crack is possible at the expense of postbuckling deformation of this structure and prebuckling deformation of the next structure when there are no damages in the vicinity of the crack tip. When the necessary criterion is realized, catastrophic fracture of the initial system takes place.

3. Sufficient fracture criterion at the generalized stress state

In order to describe a stress-strain state in the vicinity of the crack tip, we make use of the Leonov-Panasyuk-Dugdale model [7, 8]. Introduce Cartesian coordinate system Oxy with the origin at the crack tip, the Ox axis being directed along the crack axis. If solutions for stresses on the sharp crack continuation $y=0$ are used in the continual model through SIFs K_{I} and K_{II}, then the following relation for the linear problem to an accuracy of magnitudes of the highest infinitesimal order can be written

$$\sigma_y(x,0) \approx \frac{K_{\mathrm{I}}}{\sqrt{2\pi x}} + \sigma_\infty, \quad \tau_{xy}(x,0) \approx \frac{K_{\mathrm{II}}}{\sqrt{2\pi x}} + \tau_\infty, \tag{4}$$

where σ_∞ and τ_∞ are specific stresses given at infinity or on the contour of a bounded body; K_{I} and K_{II} are total SIFs, which can be given as

$$K_{\mathrm{I}} = K_{\mathrm{I}\infty} + K_{\mathrm{I}\Delta}, \quad K_{\mathrm{I}\infty} > 0, \quad K_{\mathrm{I}\Delta} < 0, \quad K_{\mathrm{II}} = K_{\mathrm{II}\infty} + K_{\mathrm{II}\Delta}, \quad K_{\mathrm{II}\infty} > 0, \quad K_{\mathrm{II}\Delta} < 0. \tag{5}$$

Here $K_{\mathrm{I}\infty}$ and $K_{\mathrm{II}\infty}$ are SIFs generated by stresses σ_∞ and τ_∞; $K_{\mathrm{I}\Delta}$ and $K_{\mathrm{II}\Delta}$ are SIFs generated by stresses σ^* and τ^* acting in the vicinity of the imaginary crack tip in the prefracture zone.

In the classical Leonov-Panasyuk-Dugdale model, an inner straight crack of length $2l_0$ is changed by an imaginary crack-cut of length $2l = 2l_0 + 2\Delta$, where Δ is the length of loaded segment or the prefracture zone length, two prefracture zones being on the continuation of the initial crack. The scheme of force loading of the right imaginary crack tip in the generalized Leonov-Panasyuk-Dugdale model is given in Fig. 4, *a* (the crack extends rectilinearly) and in Fig 4, *b* (crack path kinking). Further, the case of quasi-brittle fracture is considered when $\Delta / l_0 \ll 1$. In the classical model, only normal stresses σ^* act in the prefracture zone, shear stresses τ_∞ and τ^* are absent. Stresses σ^* and τ^* coincide with those of critical states (Fig. 2). The total SIF K_{I} can not be negative since for $K_{\mathrm{I}} < 0$, crack flanks are superimposed that is impossible physically.

In order to describe a fracture process for branching crack in the prefracture zone, we use the sufficient fracture criterion [9, 10]. In Cartesian coordinate system Oxy, the origin is at the right crack branch tip, and the Ox axis is directed along the branch (Fig. 4, *b*), the branch being at the angle θ^* to the main crack plane. Thus, the prefracture zone occupies the area along the all branch length. Denote by Δ the prefracture zone length and by a the prefracture zone width.

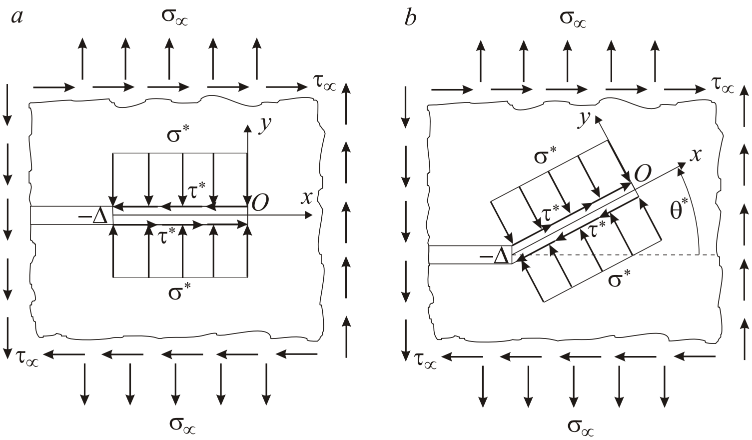

Fig. 4. The scheme of strength loading for the right imaginary crack tip in the generalized Leonov-Panasyuk-Dugdale model.

The sufficient discrete-integral criterion of quasi-brittle fracture for a sharp crack has the form

$$\frac{1}{r_0}\int_0^{r_0}\sigma_y(x,0)dx \le \sigma^*, \quad \frac{1}{r_0}\int_0^{r_0}\tau_{xy}(x,0)dx \le \tau^*, \quad x \ge 0; \tag{6}$$

$$2\upsilon(-\Delta) = \frac{\kappa+1}{G}K_{\mathrm{I}}\sqrt{\frac{\Delta}{2\pi}} \le 2\upsilon^*,\ 2u(-\Delta) = \frac{\kappa+1}{G}K_{\mathrm{II}}\sqrt{\frac{\Delta}{2\pi}} \le 2u^*,\ -\Delta \le x \le 0\,. \tag{7}$$

Here $\sigma_y(x,0)$ and $\tau_{xy}(x,0)$ are normal and tangential stresses on the continuation of a crack branch having integrable singularity; r_0 is the averaging interval; $2\upsilon = 2\upsilon(x)$ and $2u = 2u(x)$ are crack branch opening and displacement of crack branch flanks, respectively; $2\upsilon^*$ and $2u^*$ are critical crack branch opening and critical displacement of crack branch flanks, respectively; $\kappa = 3-4\nu$ for plane deformation, $\kappa = (3-\nu)/(1+\nu)$ for the plane stress state, where ν is Poisson ratio; G is the shear modulus; K_{I} and K_{II} are total SIFs in the generalized Leonov-Panasyuk-Dugdale model.

The prefracture zone length Δ is determined from solution of the fracture problem (6), (7) and the width of this zone a is found from solution of the elastic-plastic problem [11]. In Fig. 5, the plastic zone in the vicinity of the right crack tip is shaded, the zone diameter being equal to a.

Equate the plastic zone area of rectangle with sides a and Δ shown in Fig. 5. Thus, the plastic zone in the vicinity of the crack tip is approximated be a rectangular prefracture zone with the width a and length Δ. Critical parameters $2\upsilon^*$ and $2u^*$ are found from relations

$$2\upsilon^* = a^*(\varepsilon^* - \varepsilon_0), \quad 2u^* = a^*(\gamma^* - \gamma_0), \tag{8}$$

where $\tilde{\varepsilon}_0 = \sqrt{(\varepsilon_0)^2 + (\gamma_0)^2}$ and $\tilde{\varepsilon}^* = \sqrt{(\varepsilon^*)^2 + (\gamma^*)^2}$ are determined from the $\tilde{\sigma} - \tilde{\varepsilon}$ diagram (Fig. 3).

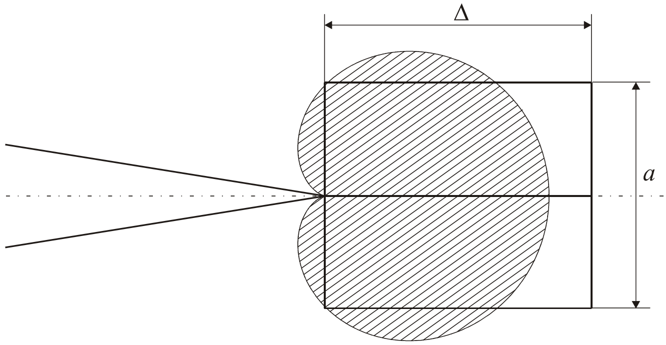

Fig. 5. Plasticity zone in the vicinity of the crack tip and prefracture zone.

The system of the first relations for the criterion (6) and (7) is equivalent to that of the second relations (6) and (7) if proportional loading takes place and stress tensors are coaxial [9, 10]. As distinct from classical fracture criteria [7, 8, 12], in the criterion in (6) and (7), restrictions are used

$$K_{\mathrm{I}} > 0, \quad K_{\mathrm{II}} \neq 0, \tag{9}$$

and the prefracture zone width is identifiable with the plasticity zone diameter at the real crack tip. Below restrictions (9) are used when deriving critical fracture parameters.

Let us elucidate how the sufficient criterion (6) and (7) is used at a crack branch. Remind that the angle θ^* has just been determined from the necessary criterion. Let a sharp inner crack of length $2l_0$ be given, material being in the initial state ahead of the crack tip, then there is no prefracture zone and its length $\Delta = 0$. At proportional loading $\tau_\infty / \sigma_\infty = c = \mathrm{const}$, the crack does not extend so long as loads $\sigma_\infty < \sigma_\infty^0$ are applied, where σ_∞^0 are critical stresses for sharp cracks obtained with the help of the necessary fracture criterion (1). When load exceeds critical stresses for the necessary criterion $\sigma_\infty > \sigma_\infty^0$, crack initiation occurs and non-elastic deformation of material in the prefracture zone begins, in this case $\Delta = \Delta(\sigma_\infty)$. Now we consider the simplest case when the prefracture zone is on the crack continuation, therefore, the length of a model crack is estimated as follows: $2l = 2l_0 + 2\Delta$. Relations (6) control conditions of the model crack initiation. Simultaneously with the prefracture occurrence, force bonds are formed in the vicinity of the model crack tip in compliance with the generalized Leonov-Panasyuk-Dugdale model (Fig. 4). Because of acting force bonds near the crack tip, steady crack growth $2l_0 < 2l < 2l^*$ takes place until a certain loading level σ_∞^* is reached, where σ_∞^* are critical stresses for sharp cracks obtained by the sufficient fracture criterion (6) and (7). In this case, $\sigma_\infty^* > \sigma_\infty^0$, $2l^* = 2l_0 + 2\Delta^*$ is the critical length of a model inner crack, $\Delta^* = \Delta(\sigma_\infty^*)$ being the critical prefracture zone length. Relations (7) control conditions for break of force bonds acting in the prefracture zone ahead of the real crack tip. When the prefracture zone length Δ coincides with the critical value Δ^*, steady crack growth is changed by unsteady one.

4. Critical fracture parameters

4.1. Straightforward crack extension ($\theta^* = 0$). Let us derive relations connecting critical parameters K_{I}^*, K_{II}^*, and Δ^* for a sharp crack extending rectilinearly $\theta^* = 0$ in quasi-brittle material. For critical values K_{I}^*, K_{II}^*, and Δ^*, relations (6) and (7) are transformed into equalities. For the SIF K_{I}^* and prefracture zone length Δ^*, and for the SIF K_{II}^* and prefracture zone length Δ^*, the second relations from (6) and (7) are used. After appropriate transformations, we have

$$K_{\mathrm{I}}^* = \sqrt{\frac{\pi r_0}{2}}\left(\sigma^* - \sigma_\infty^*\right), \quad K_{\mathrm{II}}^* = \sqrt{\frac{\pi r_0}{2}}\left(\tau^* - \tau_\infty^*\right); \tag{10}$$

$$\Delta^* = 8\pi\left(\frac{G}{\kappa+1}\frac{v^*}{K_{\mathrm{I}}^*}\right)^2, \quad \Delta^* = 8\pi\left(\frac{G}{\kappa+1}\frac{u^*}{K_{\mathrm{II}}^*}\right)^2. \tag{11}$$

For total SIFs K_{I} and K_{II} generated by stresses σ_∞ and τ_∞ acting at infinity, and stresses σ^* and τ^* acting on a segment $[-\Delta;\ 0]$, the following expression is valid [13]

$$K_{\mathrm{I}} = \sigma_\infty\sqrt{\pi l} - \sigma^*\sqrt{\pi l}\left[1 - \frac{2}{\pi}\arcsin\left(1 - \frac{\Delta}{l}\right)\right], \quad K_{\mathrm{II}} = \tau_\infty\sqrt{\pi l} - \tau^*\sqrt{\pi l}\left[1 - \frac{2}{\pi}\arcsin\left(1 - \frac{\Delta}{l}\right)\right]. \tag{12}$$

The first and second relations in (10)–(12) are equivalent if proportional loading takes place and tensors of stresses and deformations are coaxial. Thus, two nonlinear systems of equations (10) and (11) are obtained with respect to K_{I}^* and Δ^* or K_{II}^* and Δ^*.

We obtain estimates for the prefracture zone length Δ. Relations (12) can be essentially simplified when the length of a loaded segment is much less than the half-length of the crack, i.e., $\Delta/l \ll 1$. Since

$$\arcsin\left(1 - \frac{\Delta}{l}\right) \approx \frac{\pi}{2} - \sqrt{\frac{2\Delta}{l}},$$

then, the first relation (12) written for critical parameters is transformed into the form

$$K_{\mathrm{I}}^* = \sigma_\infty^*\sqrt{\pi l^*} - \sigma^*\sqrt{\pi l^*}\,\frac{2\sqrt{2}}{\pi}\sqrt{\frac{\Delta^*}{l^*}}, \tag{13}$$

where $l^* = l_0 + \Delta^*$. From the first relation (11) and (13) after appropriate transformations, we get the quadratic equation for the dimensionless parameter $\sqrt{\Delta/l}$

$$\left(\sqrt{\frac{\Delta^*}{l^*}}\right)^2 - \frac{\pi}{2\sqrt{2}}\frac{\sigma_\infty^*}{\sigma^*}\sqrt{\frac{\Delta^*}{l^*}} + \frac{\pi}{\kappa+1}\frac{v^*}{l^*}\frac{G}{\sigma^*} = 0.$$

Neglecting the values of the highest infinitesimal order, we get a simple expression for the lesser root of the quadratic equation

$$\sqrt{\frac{\Delta^*}{l^*}} \approx \frac{2\sqrt{2}}{\kappa+1}\frac{v^*}{l^*}\frac{G}{\sigma_\infty^*}. \tag{14}$$

If the restriction $\Delta^*/l^* \ll 1$ is not realized, a transcendental equation for determination of Δ^*/l^* is derived from relations (11) and (12). There are no specific difficulties in solving this equation if it has a positive root less than unity.

The critical SIF K_I^* of a sharp inner crack (13) can be written in the form

$$K_\mathrm{I}^* = \sigma_\infty^* \sqrt{\pi l^*}\left(1 - \frac{2\sqrt{2}}{\pi}\frac{\sigma^*}{\sigma_\infty^*}\sqrt{\frac{\Delta^*}{l^*}}\right). \qquad (15)$$

Take into consideration the first relation from (10) and equation (15), then the fracture assessment curve by the sufficient criterion is written as

$$\frac{\sigma_\infty^*}{\sigma^*} = \left(1 + \frac{4}{\pi}\sqrt{\frac{\Delta^*}{r_0}}\right)\left(1 + \sqrt{\frac{2l^*}{r_0}}\right)^{-1}. \qquad (16)$$

Thus, for the case of proportional loading and coaxial tensors of stresses and deformation, the system of two nonlinear equations (14) and (16) with respect to critical parameters Δ^* и σ_∞^* has been obtained, that describes formation of the prefracture zone and fracture assessment curve for the complicated stress state. It is obvious that for the mode II fracture, it is expedient to use the equivalent system of two nonlinear equations in virtue of equivalency of the first and second relations in (10), (11)

$$\sqrt{\frac{\Delta^*}{l^*}} \approx \frac{2\sqrt{2}}{\kappa + 1}\frac{u^*}{l^*}\frac{G}{\tau_\infty^*}, \quad \frac{\tau_\infty^*}{\tau^*} = \left(1 + \frac{4}{\pi}\sqrt{\frac{\Delta^*}{r_0}}\right)\left(1 + \sqrt{\frac{2l^*}{r_0}}\right)^{-1}. \qquad (17)$$

Compare critical loads obtained through necessary and sufficient criteria for the same crack lengths. From the necessary criterion (1), we have

$$K_\mathrm{I}^0 = \sqrt{\frac{\pi r_0}{2}}\left(\sigma^* - \sigma_\infty^0\right).$$

Taking into account that according to (12) $K_\mathrm{I}^0 = \sigma_\infty^0 \sqrt{\pi l_0}$, we get

$$\frac{\sigma_\infty^0}{\sigma^*} = \left(1 + \sqrt{\frac{2l_0}{r_0}}\right)^{-1}.$$

For the same crack lengths $l_0 = l^*$, we find

$$\frac{\sigma_\infty^*}{\sigma_\infty^0} = 1 + \frac{4}{\pi}\sqrt{\frac{\Delta^*}{r_0}}.$$

As it can be seen from the expression derived, critical loads obtained through necessary and sufficient criteria can essentially differ. Fig. 6 presents schematically steady (curve 1) and unsteady (curve 2) fragments of crack growth, as well as the fracture curve obtained through the necessary criterion [14] (curve 3). On the steady-growth fragment, new formed systems respond to increasing load since $\sigma_\infty^* > \sigma_\infty^0$ causing the crack to extend since $l^0 < l^*$.

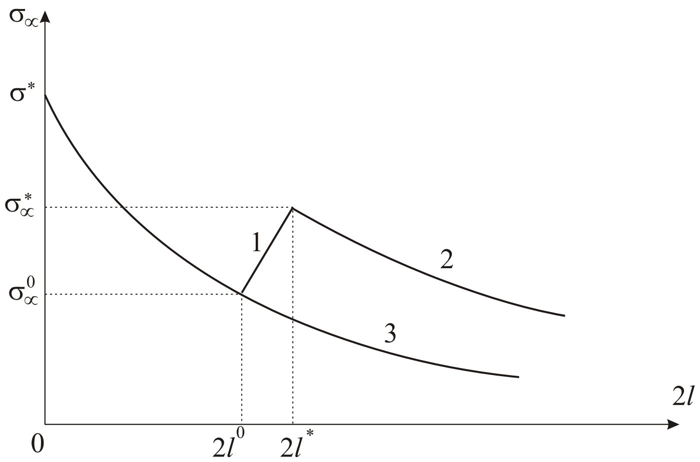

Fig. 6. Steady (curve 1) and unsteady (curve 2) segments of crack growth as well as the fracture curve obtained through necessary criterion (curve 3).

4.2. Crack kinking and branching ($\theta^* \neq 0$). In the reference literature there are no analytical solutions for stress intensity factors of the type (12) for the case of a crack with branches extending in quasi-brittle material. This generates a need for applying numerical methods. In this case, asymptotic of a stress field in the vicinity of the crack branch tip has the form

$$\sigma_y(x,0) \approx \frac{K_{\mathrm{I}}}{\sqrt{2\pi x}} + \hat{\sigma}, \quad \tau_{xy}(x,0) \approx \frac{K_{\mathrm{II}}}{\sqrt{2\pi x}} + \hat{\tau}, \tag{18}$$

where

$$\hat{\sigma} = \sigma_\infty \cos^2\theta^* - \tau_\infty \sin 2\theta^* + \ldots, \quad \hat{\tau} = \sigma_\infty \sin\theta^* \cos\theta^* + \tau_\infty \cos 2\theta^* + \ldots. \tag{19}$$

Here by dots denote singular terms stemming from crack kinking with the singularity exponent $0 < \alpha < 1/2$.

In the general case, SIFs K_{I} and K_{II} depend on $\sigma_\infty, \tau_\infty, \sigma^*, \tau^*$, l_0, Δ, and θ^*. Here σ^*, τ^*, l_0, and θ^* are constants, $\sigma_\infty, \tau_\infty$, and Δ are variable parameters, therefore, we can write $K_{\mathrm{I}} = K_{\mathrm{I}}(\hat{\sigma},\ \Delta)$ and $K_{\mathrm{II}} = K_{\mathrm{II}}(\hat{\tau},\ \Delta)$. From the first relations from (6), (7) и (18), we have for critical values

$$\hat{\sigma}^* = \sigma^* - \sqrt{\frac{2}{\pi r_0}} K_{\mathrm{I}}^*(\hat{\sigma}^*,\ \Delta^*), \quad \Delta^* = 8\pi\left(\frac{G}{\kappa+1}\frac{\upsilon^*}{K_{\mathrm{I}}^*(\hat{\sigma}^*,\ \Delta^*)}\right)^2. \tag{20}$$

In this case, the dependence $K_{\mathrm{I}}^* = K_{\mathrm{I}}^*(\hat{\sigma}^*,\ \Delta^*)$ is implicit, therefore, for solution of the system of equations (20), the following iteration procedure is proposed

$$\hat{\sigma}^*_{i+1} = \sigma^* - \sqrt{\frac{2}{\pi r_0}} K^*_{\mathrm{I}}(\hat{\sigma}^*_i,\ \Delta^*_i), \quad \Delta^*_{i+1} = 8\pi \left(\frac{G}{\kappa+1} \frac{v^*}{K^*_{\mathrm{I}}(\hat{\sigma}^*_{i+1},\ \Delta^*_i)} \right)^2,$$

where i is iteration number. From the second relations from (6) and (7), the system of equations that is equivalent to (20) is derived.

At each iteration, for given values σ_∞, and Δ, SIFs $K_{\mathrm{I}} = K_{\mathrm{I}}(\hat{\sigma},\ \Delta)$ and $K_{\mathrm{II}} = K_{\mathrm{II}}(\hat{\tau},\ \Delta)$ can be found by the finite element method from solution of the tension problem of a plate weakened with the crack with bisymmetric branches at the angles $\pm\theta^* \neq 0$.

At low angles θ^*, singularity exponent in (19) $\alpha \approx 0$. If, besides, a branch is long enough, i.e., the branch tip is far from kinking, then singular terms in (19) can be neglected, and taking into account the proportionality of the loading ($\tau / \sigma = c = \mathrm{const}$), we write

$$\hat{\sigma}^* = \sigma^*_\infty(\cos^2\theta^* - c\sin 2\theta^*), \quad \hat{\tau}^* = \sigma^*_\infty(\sin\theta^* \cos\theta^* + c\cos 2\theta^*),$$

whence the critical load $\hat{\sigma}^*$ is easily found. In the opposite case (incipient branch, the branch angle θ^* is not low), there is a need to perform additional iteration process to find the critical load $\hat{\sigma}^*$ in order to separate singular terms in (19).

Consider the particular example for a plate, the scheme of which is shown in Fig. 7, *a* (compare with Fig. 1). Sizes of the plate are as follows: 400×200 mm, $l_0 = 100$ mm, and $\Delta = 1$ mm; characteristics of the material are: Young modulus $E = 2 \times 10^5$ N/mm^2, Poisson ratio $\nu = 0.3$, the yield strength $\sigma_y = 225$ N/мм^2. On the outer contour of the plate, uniformly distributed stresses $\sigma_\infty = 25$ N/mm^2 are given, normal compressing and tangential stresses $\sigma^* = 225$ N/mm^2 and $\tau^* = 30$ N/mm^2 act at branches. In virtue of the presence of two symmetry planes, the calculation area represents the upper right quarter of the plate. Branch angles $\theta^* = 15^\circ$, 30°, and 45°. The calculation area was divided into 23000 eight-node rectangle and triangle elements, mesh sizes being decreased in the vicinity of the crack branch tip with the coefficient 100:1. Fig. 7, b demonstrates the mesh fragment in the vicinity of a branch.

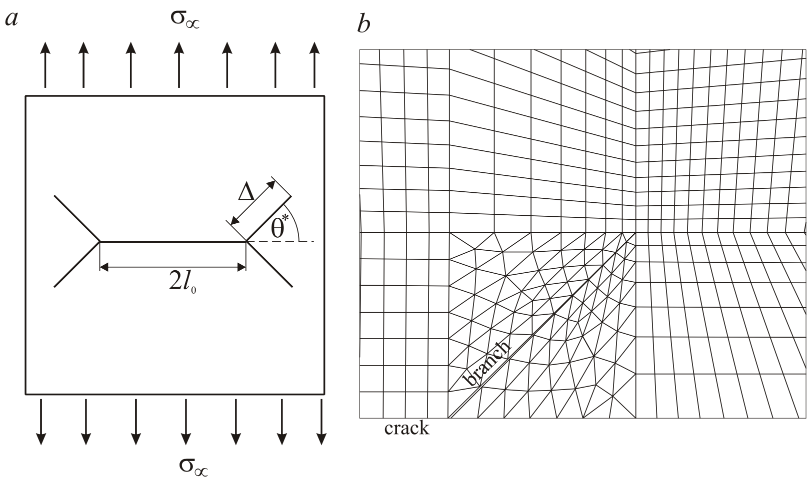

Fig. 7. The scheme of loading of a plate with crack (*a*), mesh fragment in the vicinity of the branch tip (b).

Table 1

θ^*	K_{I}	K_{II}
15°	415,71	57,042
30°	387,94	161,03
45°	305,33	258,77

The calculation results are listed in the Table 1. In order to obtain SIFs K_{I} and K_{II}, Rice *J*-integral [11] was used, which is connected to K_{I} and K_{II} by the relation $EJ = K_{\mathrm{I}}^2 + K_{\mathrm{II}}^2$ as a first approximation of the plane stress state. The contour integral was transformed into the integral over the area and integration was performed with respect to elements nearest to the branch tip. Separation into modes I and II was performed by the DeLorenzi method [15]. Solution of the same problem but for stress-free crack contour was compared with that obtained for the infinite plane in the case of uniaxial tension by the method of singular integral equations [13, p. 68]. The distinction between solutions through K_{I} is 14%, and that trough K_{II} is 10%. This can be taken as an acceptable result if it is considered that solutions for the finite and infinite areas are compared. These results can be improved if more dense mesh is used.

Summary

In the work, the critical fracture parameters (prefracture zone lengths and loads) for branched cracks in quasi-brittle materials have been obtained using the modified Leonov-Panasyuk-Dugdale model when the prefracture zones occupy rectangles located along the crack branches. The width of each prefracture zone is determined from solution of the simplest the plasticity problem for the area near the crack tip. In formulating the fracture criteria, the simplest approximations of fracture assessment curves for real materials were used. The proposed modification of the Leonov-Panasyuk-Dugdale model allows estimation of a critical crack opening and critical displacement of crack flanks for branches to be made. For deriving simple expressions for critical fracture parameters, SIFs for straightforward crack are used, and SIFs for crack with branches are used when normal and shear stresses modeling a plasticity zone are given for the branch. In order to construct SIFs for the crack with branches, the problem of uniaxial tension of a plate with the crack having bisymmetric branch has been solved. Iteration procedure is proposed that allows one to obtain critical values of SIFs by the mixed mode depending on the crack branch angle. This procedure is implemented just as for short branches when the prefracture zone occupies the all branch length, so for long branches when the branch length essentially exceeds the prefracture zone length.

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