Analysis of Multiple Cracks in Thin Coating on Orthotropic Substrate

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Abstract. The analysis addresses a typical failure development pattern consisting of a system of multiple surface cracks leading to and branching along or near the interface between the coating and the base material. The process is driven by thermal stresses. Due to the high temperature gradients during the fabrication process, usually a net of surface cracks develops, which gives the appearance of a granular structure of the surface. A periodic array of parallel surface cracks is assumed. A ''unit cell'' or single cracked segment attached to the substrate is analyzed instead by assuming the channel cracks are spaced more or less uniformly and perfectly aligned in parallel in the transverse direction of the coating. The problem is solved using both FEM combined with the reciprocal theorem and the technique of distributed dislocations. Existing semi-analytical solution for singularities in anisotropic trimaterials is applied.

Introduction

This is a matter of evidence that, in layered materials, multiple mode I surface cracks, often driven by thermal stresses, develop from a free surface and terminate at the interface. These cracks usually exhibit regular spacing that is of the same order of magnitude as the thickness of the fractured layer. It was found that there exists a ratio of fracture spacing to the layer thickness when the normal stress acting perpendicular to the fractures near the free surface changes from tensile to compressive [1], thus prohibiting inception of further cracks unless they are driven by mechanisms other than a pure extension, or there are flaws that significantly perturb the local stress field between the fractures. Under increasing applied strain the existing fractures continue to open to accommodate the applied strain. This phenomenon is called *fracture saturation.*

The aim of the paper is to generalize the analysis to orthotropic and transversally isotropic materials and to evaluate the generalized stress intensity factor (GSIF) H as a function of the ratio of fracture spacing to the layer thickness, especially near the critical value of this ratio, taking into account the presence of residual stresses. Further, there will be examined the competition between penetration and debond for periodically distributed edge cracks especially near the critical value of the ratio of fracture spacing to the layer thickness.

Analysis

Dislocation technique approach. Edge cracks in the surface layer are modeled by distributed dislocation technique. Choi and Earmme [2] used the method of analytic continuation and the Schwarz-Neumann alternating technique to obtain a solution for dislocation in an anisotropic trimaterial. The solution is expressed in terms of infinite series for the analytic functions from which the elastic field can be derived. Assume, that the regions $y \geq 0$ and $0 \geq y \geq -h$ are occupied by material 1 and 2, respectively. Both materials are perfectly bonded along the interface $y = 0$. The superscripts I, II refer to the material 1 and 2 respectively. The following relations for potentials were obtained

$$\Phi_\alpha(z)=\begin{cases}\sum_{n=1}^{\infty}\left[\Phi_\alpha^n(z)+\sum_\beta\left(\mathbf{M}^{II}\bar{\mathbf{L}}^{II}\right)_{\alpha\beta}\bar{\Phi}_\beta^n\left(z-p_\alpha^{II}h+\bar{p}_\beta^{II}h\right)\right], & z\in 2,\\ C_{\alpha\beta}\Phi_{\beta o}(z)+\sum_\beta\sum_\gamma\left(\mathbf{CM}^{II}\right)_{\alpha\beta}\bar{L}_{\beta\gamma}^{II}\sum_{n=1}^{\infty}\bar{\Phi}_\gamma^n\left(z-p_\beta^{II}h+\bar{p}_\gamma^{II}h\right), & z\in 1,\end{cases} \tag{1}$$

in which the recurrence formula for $\Phi_\alpha^n(z)$ is

$$\Phi_\alpha^{n+1}(z)=\begin{cases}\Phi_{\alpha o}(z)+\sum_\beta G_{\alpha\beta}\bar{\Phi}_{\beta o}(z), & \text{if } n=0\\ \sum_\beta G_{\alpha\beta}\left(\bar{\mathbf{M}}^{II}\mathbf{L}^{II}\right)_{\beta\gamma}\Phi_\gamma^n\left(z-\bar{p}_\beta^{II}h+p_\gamma^{II}h\right), & \text{if } n=1,2,3..\end{cases} \tag{2}$$

where the potential function for an isolated dislocation located at the point (x_o,y_o) in an infinite homogeneous anisotropic medium is

$$\Phi_{\alpha o}(z)=q_\alpha\ln\left(z-\varsigma_\alpha\right), \tag{3}$$

where $\varsigma_\alpha=x_o+p_\alpha y_o$, α=1,..3, $\mathbf{q}=\frac{1}{2\pi}\mathbf{M}\left(\mathbf{B}+\bar{\mathbf{B}}\right)^{-1}\mathbf{b}=\frac{1}{2\pi}\mathbf{Fb}$ where $\mathbf{b}=[b_x,0]^T$ is the Burgers vector, $\mathbf{M}=\mathbf{L}^{-1}$, $\mathbf{B}=i\mathbf{AM}$, p_α are three distinct complex numbers with positive imaginary parts, which are obtained as the roots of the characteristic equation

$$\det\left[c_{i1k1}+p\left(c_{i1k2}+c_{i2k1}\right)+p^2c_{i2k2}\right]=0, \tag{4}$$

where c_{ijkl} is the tensor of elastic constants. The matrices **A** and **L** are given by

$$L_{i\alpha}=A_{k\alpha}\left(c_{i2k1}+p_\alpha c_{i2k2}\right), \tag{5}$$

where $A_{k\alpha}$ denotes the eigenvector corresponding to the eigenvalue p_α above. The matrices **C** and **G** in Eqs. 1 and 2 are then defined as

$$\begin{aligned}\mathbf{C}&=i\mathbf{M}^I\mathbf{H}^{-1}\left(\mathbf{A}^{II}\mathbf{M}^{II}-\bar{\mathbf{A}}^{II}\bar{\mathbf{M}}^{II}\right)\mathbf{L}^{II}=\mathbf{M}^I\mathbf{H}^{-1}\left(\mathbf{B}^{II}+\bar{\mathbf{B}}^{II}\right)\mathbf{L}^{II},\\ \mathbf{G}&=-i\mathbf{M}^{II}\bar{\mathbf{H}}^{-1}\left(\bar{\mathbf{A}}^{II}\bar{\mathbf{M}}^{II}-\bar{\mathbf{A}}^I\bar{\mathbf{M}}^I\right)\bar{\mathbf{L}}^{II}=\mathbf{M}^{II}\bar{\mathbf{H}}^{-1}\left(\bar{\mathbf{B}}^{II}-\bar{\mathbf{B}}^I\right)\bar{\mathbf{L}}^{II}.\end{aligned} \tag{6}$$

The preceding relations were already used for modelling of a single edge crack, cf. [3]. In the case of modelling a periodic array of edge cracks illustrated in Fig. 1a, we start with a periodic array of dislocations, see Fig. 1b. Assume a periodic array of edge dislocations distributed along the line $y=y_0$. The potentials $\Phi_{\alpha o}(z)$ corresponding to the homogenous solution then read

$$\Phi_{\alpha o}(z)=\sum_{m=-\infty}^{\infty}q_\alpha^{II}\ln\left(z-(md+p_\alpha^{II}y_0)\right). \tag{7}$$

The potential $\Phi_\alpha(z)$ for $z\in 2$ can be written in the form

$$\Phi_\alpha(z) = \Phi_{\alpha o}(z) + \Phi_\alpha^{ns}(z),$$
$$\Phi_\alpha^{ns}(z) = \sum_\beta G_{\alpha\beta}\bar{\Phi}_{\beta o}(z) + \sum_{n=2}^{\infty}\Phi_\alpha^n(z) + \sum_{n=1}^{\infty}\sum_\beta\left(\mathbf{M}^{II}\bar{\mathbf{L}}^{II}\right)_{\alpha\beta}\bar{\Phi}_\beta^n\left(z - p_\alpha^{II}h + \bar{p}_\beta^{II}h\right). \tag{8}$$

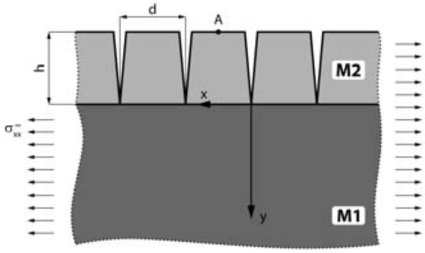

Figure 1a: Scheme of an array of periodically distributed edge cracks

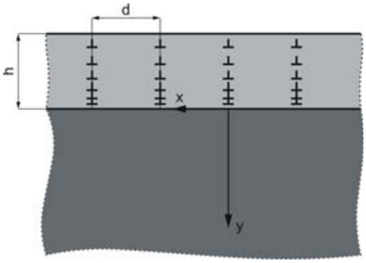

Figure 1b: Scheme of periodically distributed dislocation arrays

The stress component $\sigma_{xx}^{II}(x,y)$ then follows as

$$\sigma_{xx}^{II}(x,y) = -2\,\mathrm{Re}\left[L_{11}^{II}p_1^{II}\Phi_1'\left(z_1^{II}\right) + L_{12}^{II}p_2^{II}\Phi_2'\left(z_2^{II}\right)\right] =$$
$$= -2\,\mathrm{Re}\left\{L_{11}^{II}p_1^{II}\sum_{m=-\infty}^{\infty}\frac{q_1^{II}}{z_1^{II} - (md + p_1^{II}y_0)} + L_{12}^{b}p_2^{b}\sum_{m=-\infty}^{\infty}\frac{q_2^{II}}{z_2^{II} - (md + p_2^{II}y_0)}\right\} + \sigma_{xx}^{II,ns}(x,y), \tag{9}$$

where

$$\sigma_{xx}^{II,ns}(x,y) = -2\,\mathrm{Re}\left\{L_{11}^{II}p_1^{II}\sum_{m=-\infty}^{\infty}\left(\Phi_{1,m}^{II,ns}(z_1^{II})\right)' + L_{12}^{II}p_2^{II}\sum_{m=-\infty}^{\infty}\left(\Phi_{2,m}^{II,ns}(z_2^{II})\right)'\right\}. \tag{10}$$

Using the formula $\sum_{k=-\infty}^{\infty}\frac{1}{z+k} = \pi\cot(\pi z)$ we can carry out the summation in Eq. 9 and get

$$\sigma_{xx}^{II}(x,y) = -\frac{2\pi}{d}\mathrm{Re}\left\{L_{11}^{b}p_1^{b}q_1^{b}\cot\left(\frac{\pi}{d}(z_1^{II} - p_1^{II}y_0)\right) + L_{12}^{II}p_2^{II}q_2^{II}\cot\left(\frac{\pi}{d}(z_2^{II} - p_2^{II}y_0)\right)\right\} + \sigma_{xx}^{II,ns}(x,y) \tag{11}$$

Similarly, the potential $\Phi_{\alpha o}(z)$ in Eq. 7 can be written as $\Phi_{\alpha o}(z) = \frac{\pi}{d}q_\alpha^{II}\cot\left(\frac{\pi}{d}(z - p_\alpha^{II}y_0)\right)$. By substituting this result into the recurrence formula for $\Phi_\alpha^n(z)$, the infinite series in Eq. 10 are replaced by their summations.

Now, the edge cracks are simulated by distributed dislocation arrays with the dislocation density $db_x = f(y_0)dy_0$ or $dq_\alpha^{II}(y_0) = \frac{1}{2\pi}F_{\alpha 1}^{II}f(y_0)dy_0$. If the crack faces are traction free, the following condition must be fulfilled:

$$\frac{1}{d}\mathrm{Re}\left\{L_{11}^{II}p_1^{II}F_{11}^{II}\int_{-h}^{0}f(y_0)\cot\left(\frac{\pi}{d}(p_1^{II}y - p_1^{II}y_0)\right)dy_0 + L_{12}^{II}p_2^{II}F_{11}^{b}\int_{-h}^{0}f(y_0)\cot\left(\frac{\pi}{d}(p_2^{II}y - p_2^{II}y_0)\right)dy_0\right\} = \tag{12}$$
$$= \sigma_{xx}^{II,ns}(0,y) + \sigma_{xx}^{\infty}(0,y),$$

where $\sigma_{xx}^{\infty}(0,y)$ is the stress acting in the layer if the cracks were absent. Observe that due to symmetry of loading the shear tractions are zero. The substitution

$$t_0=\frac{1}{2}\left(\sin\left(\frac{\pi p_\alpha^{II}}{d}(y+h)\right)-\sin\left(\frac{\pi p_\alpha^{II}}{d}y\right)\right)s_0+\frac{1}{2}\left(-\sin\left(\frac{\pi p_\alpha^{II}}{d}(y+h)\right)-\sin\left(\frac{\pi p_\alpha^{II}}{d}y\right)+2t\right) \tag{13}$$

allows to rewrite Eq. 12 into the form

$$0=\frac{d}{\pi^2}\mathrm{Re}\left\{\frac{L_{11}^{II}F_{11}^{II}}{p_1^{II}}\int_{-1}^{1}\frac{f^{**}(s_0)}{s_1-s_0}ds_0+\frac{L_{12}^{II}F_{11}^{II}}{p_2^{II}}\int_{-1}^{1}\frac{f^{**}(s_0)}{s_2-s_0}dt_0\right\}+\sigma_{xx}^{II,ns}(0,s)+\sigma_{xx}^{\infty}(0,s), \tag{14}$$

where

$$s_\alpha=\frac{\sin\left(\frac{\pi p_\alpha^{II}}{d}(y+h)\right)+\sin\left(\frac{\pi p_\alpha^{II}}{d}y\right)}{\sin\left(\frac{\pi p_\alpha^{II}}{d}(y+h)\right)-\sin\left(\frac{\pi p_\alpha^{II}}{d}y\right)}.$$

The dislocation density is sought in the form $f^{**}(s)=(1-s)^{-(1-\delta_1)}(1+s)^{1-\delta_1}g(s)$, where $g(s)$ is a bounded function, and 1-δ_1 is the stress singularity exponent. This choice means that $f^{**}(-1)$ must vanish, i.e. that crack faces at the mouth are forced to be parallel and the solution is over-constrained. This incorrect end-point behaviour at the crack mouth has a negligible effect on the calculated stress intensity factor, however, it can influence the stress σ_{xx} $(x,-h)$ in between the edge cracks. The integral equation may be solved using the Gauss-Jacobi quadrature. The function g(s) is sought in the form of linear combination of Jacobi polynomials $P_n^{(\lambda,-\lambda)}(s)$

$$g(s)=\sum_{n=0}^{\infty}c_nP_n^{(\lambda,-\lambda)}(s)\cong\sum_{n=0}^{N_B}c_nP_n^{(\lambda,-\lambda)}(s),$$

which leads to the system of linear equations

$$\sigma_{xx}^{\infty}(0,s_i)-\sum_{\alpha=1}^{2}\mathrm{Re}\left\{\frac{L_{1\alpha}^{II}F_{11}^{II}}{p_\alpha^{II}}\right\}\sum_{n=0}^{N_B}c_n\left[\cot(-\pi\lambda)(1-s_{\alpha,i})^{-\lambda}(1+s_{\alpha,i})^{\lambda}P_n^{(-\lambda,\lambda)}(s_{\alpha,i})-\right.$$
$$\left.-\frac{\Gamma(-\lambda)\Gamma(n+\lambda+1)}{\Gamma(n+1)}\mathrm{F}\left(n+1,-n;1+\lambda;\frac{1-s_{\alpha,i}}{2}\right)\right]+\sum_{n=0}^{N_B}c_nd_n(s_i)\frac{2}{2n+1}\frac{\Gamma(n-\lambda+1)\Gamma(n+\lambda+1)}{n!\Gamma(n+1)}=0,$$

where $N_B\le N_f$, $s_{\alpha,i}=\dfrac{\sin\left(\frac{\pi p_\alpha^{II}}{d}(y_i+h)\right)+\sin\left(\frac{\pi p_\alpha^{II}}{d}y_i\right)}{\sin\left(\frac{\pi p_\alpha^{II}}{d}(y_i+h)\right)-\sin\left(\frac{\pi p_\alpha^{II}}{d}y_i\right)}$, $s_i=\dfrac{2}{h}y_i+1$,

where $y_i\in(-h,0)$ are collocations points at the crack faces where the stress equilibrium is controlled, F(n_1,n_2;n_3;x) stands for the hypergeometric function, $\Gamma(n)$ is the Gamma function and i= 0.1..N_B –1. Once the dislocation density is found, the potentials $\Phi_\alpha(z)$ are evaluated and the displacement and stress fields can be obtained via formulas

$$u_i=2\,\mathrm{Re}\left[\sum_{\alpha=1}^{3}A_{i\alpha}\Phi_\alpha(z_\alpha)\right],\quad \sigma_{2i}=2\,\mathrm{Re}\left[\sum_{\alpha=1}^{3}L_{i\alpha}\Phi'_\alpha(z_\alpha)\right],\quad \sigma_{1i}=-2\,\mathrm{Re}\left[\sum_{\alpha=1}^{3}L_{i\alpha}p_\alpha\Phi'_\alpha(z_\alpha)\right] \tag{15}$$

GSIF H is calculated using the function-theoretic methods, see e.g. [3].

FEM and Ψ-integral approach. In the absence of body forces the reciprocal theorem states that the following integral is path independent

$$\Psi(\mathbf{u},\mathbf{v}) = \int_{\Gamma}\left[\sigma_{ij}(\mathbf{u})n_i\, v_j - \sigma_{ij}(\mathbf{v})n_i u_j\right]\mathrm{d}s\,, \tag{16}$$

where Γ is any contour surrounding the crack tip and $\mathbf{u}$, $\mathbf{v}$ are two admissible displacement fields. If the following displacement fields are considered $\mathbf{u} = \mathcal{U}_i(x) = r^{\delta_i}\mathbf{u}_i(\theta)$, $\mathbf{v} = \mathcal{U}_j(x) = r^{\delta_j}\mathbf{u}_j(\theta)$, one can show that the contour integral Ψ is equal to zero for $-\delta_i \neq \delta_j$ and non-zero if $-\delta_i = \delta_j$. Since the basis function corresponding to coefficient $f_1 = H$ in the asymptotic expansion for $\mathbf{u}$ is $r^{\delta_1}\mathbf{u}_1(\theta)$, due to the former "orthogonality" conditions it holds

$$\Psi\left(\mathbf{u}, r^{-\delta_1}\mathbf{u}_{-1}\right) = \sum_{i=1}^{\infty} f_i \Psi\left(r^{\delta_i}\mathbf{u}_i, r^{-\delta_1}\mathbf{u}_{-1}\right) = f_1\Psi\left(r^{\delta_1}\mathbf{u}_1, r^{-\delta_1}\mathbf{u}_{-1}\right). \tag{17}$$

Thus, the GSIF $H = f_1$ can be computed as follows:

$$H = \frac{\Psi\left(\mathbf{u}, r^{-(1-\lambda_1)}\mathbf{u}_{-1}\right)}{\Psi\left(r^{1-\lambda_1}\mathbf{u}_1, r^{-(1-\lambda_1)}\mathbf{u}_{-1}\right)}. \tag{18}$$

Since the exact solution $\mathbf{u}$ is not known, a finite element solution $\mathbf{u}^h$ can be used as an approximation for $\mathbf{u}$ so to obtain an approximation for H, [3]. Due to the path independence, the Ψ-integral standing in the denominator of Eq. (18) is evaluated along an infinitesimal path that shrinks to the crack tip. To this end, the asymptotic stress field near the crack tip modelled as a continuous distribution of dislocations with density function, $f_k(y_o) = Hv_k(-y_o)^{\delta_1 - 1}, y_o < 0$ (v_k is a corresponding eigenvector) can be used. Observing that the problem is linear, the results may be applied to any combination of mechanical, thermal and residual stresses by using superposition.

The competition between penetration and debond for periodically distributed edge cracks. A necessary condition for a crack to deflect along the interface is $\Gamma_i(\psi)/\Gamma_p < G_d/G_p$, where $\Gamma_i(\psi)$ is the interface toughness at a phase angle of loading, ψ, Γ_p is the toughness of the next layer, G_d is the energy release rate for a crack deflected at the interface and G_p is the energy release rate for a penetrating crack. Deflections of periodically distributed edge cracks along the interface would increase the critical value of the ratio of fracture spacing to the layer thickness. Consider a perturbation of the domain Ω with periodically distributed edge cracks impinging the interface. The perturbation of *each of the edge cracks* is a deflected (double) crack extension of length a_d or penetrating crack extension of length a_p with the small perturbation parameter ε defined as $\tau = a/L \ll 1$, $a = a_p, a_d$, where L is the characteristic length of Ω. A second scale to the problem can be introduced, represented by the scaled-up coordinates $\rho = r/\tau$ which provides a zoomed-in view into the region surrounding the crack. The displacement $\mathbf{U}^\tau$ of the perturbed elasticity problem due to the crack extension can now be expressed in terms of the regular coordinates r,θ and the scaled-up coordinate ρ,θ as $\mathbf{U}^\tau(r,\theta) = \mathbf{U}^\tau(\tau\rho,\theta) = \mathbf{V}^\tau(\rho,\theta)$, where the definition of the function $\mathbf{V}^\tau$ has been introduced, simply by a change of variable from r to ρ. Consider now the asymptotic expansion for $\mathbf{U}^\tau$ (which is also known as the ''outer expansion'') and for $\mathbf{V}^\tau$ (which is also known as the ''inner expansion''). Outer and the inner asymptotic expansions read

$$\mathbf{U}^{\tau}\left(r=\tau\rho,\theta\right)=H\tau^{\delta_1}\left[\rho^{\delta_1}\mathbf{u}_{\;1}\left(\theta\right)+K_{1d(p)}\rho^{-\delta_1}\mathbf{u}_{\;-1}\left(\theta\right)+...\right]+...=\mathbf{V}^{\tau}\left(\rho,\theta\right) \quad (19)$$

The determination of the coefficients $K_{1d(p)}$ proceeds in a similar fashion as for H. $K_{1d(p)}$ are calculated in the inner domain whose remote boundary $\partial\Omega_{in}$ is subjected to the boundary condition $\mathbf{U}\big|_{\partial\Omega_{in}}=\rho^{\delta_1}\mathbf{u}_{\;1}\left(\theta\right)$

$$K_{1d(p)}=\frac{\Psi\left(\mathcal{V}_1^h\left(y\right),\rho^{\delta_1}\mathbf{u}_1\right)}{\Psi\left(\rho^{-\delta_1}\mathbf{u}_{-1},\rho^{\delta_1}\mathbf{u}_1\right)},\ \mathcal{V}_1^h\text{-FEM approximation to }\mathbf{V}^{\tau}. \quad (20)$$

The incremental energy release rate (ERR) $G_{d(p)}$ is defined as

$$\begin{aligned}G_{d(p)}&=-\frac{\delta W}{\tau_{d(p)}L}=-\frac{W^{\tau}-W^0}{\tau_{d(p)}L}=-\frac{1}{2\tau_{d(p)}L}\int_{\Gamma}\left(\sigma_{kl}\left(\mathbf{U}^{\tau}\right)n_kU_l^0-\sigma_{kl}\left(\mathbf{U}^0\right)n_kU_l^{\tau}\right)ds=\\&=-\frac{1}{2\tau_{d(p)}L}\Psi\left(\mathbf{U}^{\tau},\mathbf{U}^0\right)=-\frac{1}{2\tau_{d(p)}L}\Psi\left(H\tau_{d(p)}^{\delta_1}\left[\rho^{\delta_1}\mathbf{u}_{\;1}\left(\theta\right)+K_{1d(p)}\rho^{-\delta_1}\mathbf{u}_{\;-1}\left(\theta\right)+..\right],\right.\\&\left.H\tau_{d(p)}^{\delta_1}\rho^{\delta_1}\mathbf{u}_1\left(\theta\right)+..\right)=\frac{1}{2L}H^2K_{1d(p)}\tau_{d(p)}^{2\delta_1-1}\Psi\left(\rho^{\delta_1}\mathbf{u}_1\left(\theta\right),\rho^{-\delta_1}\mathbf{u}_{\;-1}\left(\theta\right)\right)+..,\end{aligned} \quad (21)$$

where $\tau_{d(p)}=a_{d(p)}/L$. Observe, that line Γ is any contour surrounding the crack tip and the crack increment and starting and finishing on the stress-free faces of he primary crack. Among others, the crack extension faces along a_p or a_d respectively, form an admissible contour which allows to rewrite Eq. 21 as a work done along $a_{d(p)}$ and leads to the classical virtual crack closure method

$$\begin{aligned}G_{d(p)}&=-\frac{\delta W}{\tau_{d(p)}L}=-\frac{W^{\tau}-W^0}{\tau_{d(p)}L}=-\frac{1}{2a_{d(p)}}\int_{a_{d(p)}}\left(\sigma_{kl}\left(\mathbf{U}^{\tau}\right)n_kU_l^0-\sigma_{kl}\left(\mathbf{U}^0\right)n_kU_l^{\tau}\right)ds=\\&=\frac{1}{2a_{d(p)}}\int_{a_{d(p)}}\sigma_{kl}\left(\mathbf{U}^0\right)n_kU_l^{\tau}\,ds=\frac{1}{2a_{d(p)}}\int_0^{a_{d(p)}}\sigma_{kl}\left(\mathbf{U}^0\right)n_k\Delta U_l^{\tau}ds,\end{aligned} \quad (22)$$

where the integral along $a_{d(p)}$ means along two faces $a_{d(p)}^+$ and $a_{d(p)}^-$ and ΔU^{τ} denotes $\Delta U_l^{\tau}=\left(U_l^{\tau}\right)^+-\left(U_l^{\tau}\right)^-$ where the sign + or – refer to upper or lower crack face. The expression (22) is rather difficult to handle numerically since the singularities govern the behaviour along $a_{d(p)}$. Nevertheless, it offers an idea to calculate the fracture mode mixity based upon the energy release rate (ERR). The ratio of the debonding to the penetrating ERR follows from Eq. 21 as

$$\frac{G_d}{G_p}=\frac{K_{1d}}{K_{1p}}\left(\frac{a_d}{a_p}\right)^{2\delta_1-1}. \quad (23)$$

The fracture mode mixity based on the stress intensity factor (SIF) concept is usually represented by the so-called local phase angle ψ_K defined by $K=K_1+iK_2=\left|K\right|e^{i\psi_K}$ where K is the complex stress intensity factor (SIF), associated to a reference length l according to the proposal by Rice [4]

The ERR based fracture mode mixity originally results from the application of the virtual crack closure method. Consider a small but finite length a_d of a virtual crack extension along the interface. The energy release rate (ERR) associated to this crack extent is

$$G_d(a_d) = G_{dI}(a_d) + G_{dII}(a_d), \tag{24}$$

where

$$G_{dI}(a_d) = \frac{1}{2a_d}\int_0^{a_d} \sigma_{22}(s,0)\Delta u_2(a_d - s)ds, \quad G_{dII}(a_d) = \frac{1}{2a_d}\int_0^{a_d} \sigma_{12}(s,0)\Delta u_1(a_d - s)ds. \tag{25}$$

The Mode I component G_{dI} corresponds to the energy released by normal stresses acting through crack face opening displacements, and Mode II component G_{dII} corresponds to the energy released by shear stresses acting through crack face sliding displacements. The energetic mode mixity G_{dI}/G_{dII} for interface crack depends on a_d . The associated phase angle ψ_G is defined as

$$\tan^2\psi_G = \frac{G_{dII}(a_d)}{G_{dI}(a_d)}, \quad 0 \le \psi_G \le \frac{\pi}{2}. \tag{26}$$

Instead of Eqs. 25, the concept of Ψ- integral can be applied for to evaluate the phase angle ψ_G. First observe that Eq. 25 can be written in the form

$$\begin{aligned} G_d &= -\frac{1}{2a_d}\int_{a_d}\left(\sigma_{kl}(\mathbf{U}^\tau)n_k U_l^0 - \sigma_{kl}(\mathbf{U}^0)n_k U_l^\tau\right)ds = \\ &= \underbrace{-\frac{1}{2a_d}\int_{a_d}\left(\sigma_{22}(\mathbf{U}^\tau)n_2 U_2^0 - \sigma_{22}(\mathbf{U}^0)n_2 U_2^\tau\right)ds}_{G_{dI}} \underbrace{-\frac{1}{2a_d}\int_{a_d}\left(\sigma_{21}(\mathbf{U}^\tau)n_2 U_1^0 - \sigma_{21}(\mathbf{U}^0)n_2 U_1^\tau\right)ds}_{G_{dII}}. \end{aligned} \tag{27}$$

On the other side, assume any contour Γ surrounding the crack tip and write

$$\begin{aligned} G_d &= -\frac{1}{2a_d}\int_{\Gamma}\left(\sigma_{kl}(\mathbf{U}^\tau)n_k U_l^0 - \sigma_{kl}(\mathbf{U}^0)n_k U_l^\tau\right)ds = -\frac{1}{2a_d}\int_{\Gamma}\left(\sigma_{kl}(\mathbf{U}^\tau)n_k\delta_{lj}U_j^0 - \sigma_{kl}(\mathbf{U}^0)n_k\delta_{lj}U_j^\tau\right)ds = \\ &= \underbrace{-\frac{1}{2a_d}\int_{\Gamma}\left(\sigma_{kl}(\mathbf{U}^\tau)n_k n_l n_j U_j^0 - \sigma_{kl}(\mathbf{U}^0)n_k n_l n_j U_j^\tau\right)ds}_{G_{dI}} \underbrace{-\frac{1}{2a_d}\int_{\Gamma}\left(\sigma_{kl}(\mathbf{U}^\tau)n_k t_l t_j U_j^0 - \sigma_{kl}(\mathbf{U}^0)n_k t_l t_j U_j^\tau\right)ds}_{G_{dII}}, \end{aligned} \tag{28}$$

where t_l is he unit tangential vector of Γ. Thus, the ERR based phase angle ψ_G for deflected crack can be calculated by substituting for G_{dI} and G_{dII} from Eq. 28 to Eq. 26. Note that the ERR and the SIF based measures of mode mixity for an interface crack, phase angle ψ_G and ψ_K, are related by [5]

$$\cos(2\psi_G) = \sqrt{\frac{\sinh(2\pi\varepsilon)}{2\pi\varepsilon(1+4\varepsilon^2)}}\cos\left[2\psi_K + 2\varepsilon\ln\frac{a_d}{2L} + \arg\left[\frac{\Gamma(1/2+i\varepsilon)}{\Gamma(1+i\varepsilon)}\right] - \arctan(2\varepsilon)\right], \tag{29}$$

$\varepsilon -$ oscillation index of the interface crack,

with $\Gamma(.)$ being the gamma function.

Numerical results

Numerical calculations were specialized to aligned orthotropic materials. The materials were characterized by two dimensionless elastic parameters γ and ρ $\gamma = s_{11}/s_{22}$, $\rho = (2s_{12}+s_{66})/\sqrt{s_{11}s_{22}}$, where s_{ij} are the material compliances and defined in the conventional fashion. The relative stiffness between these materials – coating and substrate – is measured by the two generalized Dundurs parameters α and β [6]. The developed residual stresses are characterized by the temperature change ΔT between the processing temperature T_p and current temperature T, the thermal expansion ratios $\gamma_T = \alpha_{T1}/\alpha_{T2}$ for both materials, and the thermal expansion mismatch between the materials $\Delta\alpha_{T1}$.The following example refers to case when the isotropic surface layer is formed by ZrO_2 with thickness of 13 μm deposited on Al_2O_3 substrate of thickness 40 μm reinforced by aligned SiC whiskers with the volume fractions V_f = 0.1, 0.3. For V_f = 0.3, the corresponding values of parameters introduced above are as follows:
$\gamma^{II}=1, \rho^{II}=1, \gamma^{I}=0.97, \rho^{I}=1.003, \alpha=0.33, \beta=0.067, \gamma_T^{II}=1, \gamma_T^{I}=0.94, \Delta\alpha_{T1}=3.01\cdot 10^{-6}K^{-1}$. The value of the singularity exponent is δ_1= 0.56. Fig. 2a shows a distribution of the stress component σ_{xx} between edge cracks just below the free surface for different spacing to layer thickness ratios. It should be noted that only residual stresses due to the uniform temperature change ΔT = 800°C were considered, i.e. no applied loads were prescribed.

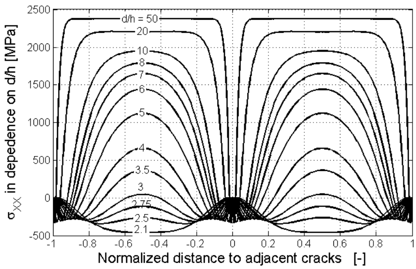

Figure 2a: Distributions of the normal stress component σ_{xx} between edge cracks for different spacing/layer thickness ratio d/h, V_f = 0.3.

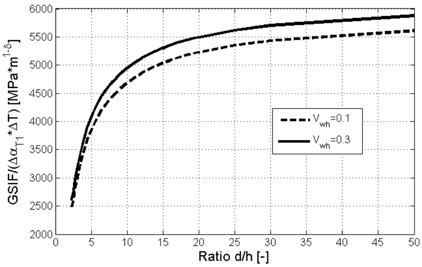

Figure 2b: GSIF normalized by the stress-free misfit strain $\Delta\alpha_{T1}\Delta T$ as a function of the spacing/layer thickness ratio d/h

It is seen from the Fig. 2a that, in the given case, the critical spacing to layer thickness ratio, i.e. the ratio at fracture saturation, is about of 2.9. Fig. 2b shows the GSIF as a function of the spacing to layer thickness ratio. Apparently, GSIF abruptly decreases when approaching the critical spacing to layer thickness ratio. Further results concerning the effect of elastic properties and delamination will be presented at the ECF17 meeting.

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