A Stedily Propagating Crack in Planar Quasicrystal with Fivefold Symmetry

E. Radi[1, a] and P.M. Mariano[2,b]

[1]Di.S.M.I., Università di Modena e Reggio Emilia, Via Amendola 2, I-42100 Reggio Emilia, Italia

[2]D.I.Ce.A, Università di Firenze, Via Santa Marta 3, I-50139 Firenze, Italia

[a]eradi@unimore.it, [b]paolo.mariano@unifi.it

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Abstract. A closed-form solution is provided for the stress, strain and velocity fields due to a planar crack steadily propagating in an elastic quasicrystal with fivefold symmetry at speed lower than the bulk wave-speeds. The case of a semi-infinite rectilinear crack loaded on its surfaces is considered. The dynamic theory of quasicrystal with inertia forces, but neglecting dissipative phonon activity, is assumed to govern the motion of the medium. Both phonon and phason stress fields display square-root singular at crack tip. The energy release rate is positive for subsonic and subRayleigh crack propagation. The limit case of a stationary crack is then recovered as the crack tip speed becomes vanishing small.

1. Introduction

Quasicrystals (QCs) are a special class of quasi-periodic alloys characterized by atomic clusters displaying incompatible symmetries with periodic tiling of atoms in space, symmetries such as the icosahedral one in three-dimensions and the penthagonal one in the plane. Quasi-periodicity in space is assured by atomic rearrangements creating and annihilating atomic clusters with symmetry different from the prevailing one. Inner degrees of freedom pertain then to each material element (a cluster of atoms). Peculiar interactions are generated: they are different from the standard stresses due to the crowding and shearing of material elements and influence even drastically the macroscopic behaviour [1]. In particular, in presence of defects such microscopic interactions contribute to both equilibrium and possible evolution of the defects themselves. A paradigmatic example is the one of cracks propagating in QCs: the effects of the atomic rearrangements influence the force driving the crack tip and, consequently, they perturb the crack path with respect to the one foreseen by neglecting the rearrangements at atomic scale (see results in [2,3]).

In the present work, we investigate steady crack propagation in an elastic QC with fivefold symmetry within the infinitesimal deformation setting, occurring at speed lower than the bulk wave velocity. A closed form solution for interactions measures, deformation, and rate fields is provided under general loading conditions. Viscous-like dissipation within material elements is neglected since the analysis is developed at a time smaller than the characteristic activation time. Macroscopic and substructural stresses display square root singularities at the crack tip. The energy release rate is evaluated for subsonic sub-Rayleigh crack propagation. Stress intensity factors are determined. The indeterminacy of the coupling coefficient between the gross deformation and the atomic rearrangements is accounted for parametrically. The method adopted for determining the closed form solution is an evolution of a previous approach used in [4,5] for linear anisotropic elasticity. The method is based on the Stroh formalism [6,7]. However, for the considered isotropic relation between phonon stress and strain, the eigenvalue problem is degenerate, namely the fundamental matrix admits two double eigenvalues and these identical roots do not have distinct eigenvectors associated with them. Therefore, the Stroh formalism has been modified in agreement with the generalization presented in [8] for degenerate orthotropic elastic materials.

Notation. Applied to the second-order tensor $\mathbf{A}$ with components A_{ij} in Cartesian axes, the divergence operator div is defined componentwise as $(\mathrm{div}\,\mathbf{A})_i = A_{ij,j}$ where $(\ )_{,j}$ denotes the partial derivative with respect to the j-th spatial coordinate and the summation over repeated indices applies. $\mathbf{A}^T$ is the transpose of $\mathbf{A}$. ∇ denotes the gradient operator, namely $(\nabla a)_i = a_{,i}$ for a scalar field a. The notation $\langle\langle a_k \rangle\rangle$ denotes the diagonal matrix diag$[a_1, a_2, a_3, a_4]$ with components a_k (k = 1, 2, 3, 4) on the principal diagonal. An overdot denotes derivatives with respect to the time t. The operators Re and Im define the real and imaginary parts of a complex number. An overbar denotes the complex conjugate of a complex number.

2. Field and constitutive equations

Neglecting body forces the equations of balance of momentum can be expressed in terms of the Cauchy stress $\boldsymbol{\sigma}$ and of the phason stress $\mathbf{S}$ as:

$$\mathrm{div}\,\boldsymbol{\sigma} = \rho\,\ddot{\mathbf{u}}, \qquad \mathrm{div}\,\mathbf{S} = c\,\dot{\mathbf{w}}. \tag{1}$$

Here $\mathbf{u}$ and $\mathbf{w}$ are the phonon and phason displacement vectors, c is a positive scalar and ρ is the mass density of the medium which is assumed to be constant. The Cauchy and phason stresses depend on the phonon and phason displacement gradients, namely $\nabla\mathbf{u}$ and $\nabla\mathbf{w}$, through the following two-dimensional linear elastic constitutive equations:

$$\boldsymbol{\sigma} = \mu\,(\nabla\mathbf{u} + \nabla\mathbf{u}^T) + \lambda\,\mathbf{I}\ \mathrm{div}\,\mathbf{u} + k_3\,(\nabla\mathbf{w} + \nabla\mathbf{w}^T - \mathbf{I}\ \mathrm{div}\,\mathbf{w})\,\mathbf{R}, \tag{2}$$

$$\mathbf{S} = k_3\,\mathbf{R}\,(\nabla\mathbf{u} + \nabla\mathbf{u}^T - \mathbf{I}\ \mathrm{div}\,\mathbf{u}) + k_1\,\nabla\mathbf{w} - k_2\,(\nabla\mathbf{w}^T - \mathbf{I}\ \mathrm{div}\,\mathbf{w}), \tag{3}$$

being $\mathbf{R} = \mathbf{e}_1 \otimes \mathbf{e}_1 - \mathbf{e}_2 \otimes \mathbf{e}_2$. The five parameters μ, λ, k_1, k_2 and k_3 define the linear constitutive response. Their values are given in [1], namely $\mu = 65\ GPa$, $\lambda = 75\ GPa$, $k_1 = 81\ GPa$, $k_2 = -42\ GPa$. Moreover, the mean value of k_3 is fixed at $0.1\ k_1$. The phonon and phason traction vectors $\mathbf{t}$ and $\mathbf{s}$ acting on a surface with outward unit normal $\mathbf{n}$ are defined as

$$\mathbf{t} = \boldsymbol{\sigma}\,\mathbf{n}, \qquad \mathbf{s} = \mathbf{S}\,\mathbf{n}. \tag{4}$$

3. Steady-state crack propagation

The problem of a plane crack propagating at constant speed v along a rectilinear path in an infinite medium is considered. Two Cartesian coordinate systems are considered, the system (0, x, y, z) is fixed in time and the other (0, x_1, x_2, x_3) is centered at the crack tip and moving with it in the x_1 direction, with the out-of-plane x_3-axis along the straight crack front. During steady-state crack propagation an arbitrary scalar or vector field $\mathbf{v}$ must obey the condition $\mathbf{v}(x_1, x_2) = \mathbf{v}(x - vt, y)$, so that $\dot{\mathbf{v}} = -v\,\mathbf{v}_{,1}$.

The term containing the velocity $\dot{\mathbf{w}}$ in (1) may be neglected since the constant c is usually very small for quasi-crystals. This term models the self equilibrated forces which originates from dissipative phenomena. By introducing the following four-dimensional vectors

$$\mathbf{p} = (\sigma_{11}, \sigma_{21}, S_{11}, S_{21}), \qquad \mathbf{q} = (\sigma_{12}, \sigma_{22}, S_{12}, S_{22}), \tag{5}$$

$$\mathbf{a} = (u_{1,1}, u_{2,1}, w_{1,1}, w_{2,1}), \qquad \mathbf{b} = (u_{1,2}, u_{2,2}, w_{1,2}, w_{2,2}), \tag{6}$$

which collect the phonon and phason stresses and components of the displacement gradient, the constitutive relations (2) and (3) may be written in the form

$$\begin{pmatrix} \mathbf{p} \\ \mathbf{q} \end{pmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^{\mathrm{T}} & \mathbf{C} \end{bmatrix} \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix}, \tag{7}$$

where

$$\mathbf{A} = \begin{bmatrix} 2\mu+\lambda & 0 & k_3 & 0 \\ 0 & \mu & 0 & k_3 \\ k_3 & 0 & k_1 & 0 \\ 0 & k_3 & 0 & k_1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & \lambda & 0 & k_3 \\ \mu & 0 & -k_3 & 0 \\ 0 & -k_3 & 0 & k_2 \\ k_3 & 0 & -k_2 & 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} \mu & 0 & -k_3 & 0 \\ 0 & 2\mu+\lambda & 0 & -k_3 \\ -k_3 & 0 & k_1 & 0 \\ 0 & -k_3 & 0 & k_1 \end{bmatrix}, \tag{8}$$

and the equilibrium equations (1) become

$$\mathbf{p}_{,1} + \mathbf{q}_{,2} = \rho v^2\, \mathbf{D}\, \mathbf{a}_{,1}, \tag{9}$$

where $\mathbf{D}$ = diag(1, 1, 0, 0) and a subscript comma denotes partial differentiation with respect to spatial coordinates. Introduction of (7) in (9) and the identity $\mathbf{b}_{,1} = \mathbf{a}_{,2}$ yields the equations of motion in terms of the phonon and phason displacements, written in the following matrix form

$$\begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix}_{,1} + \begin{bmatrix} \mathbf{Q}^{-1}(\mathbf{B}+\mathbf{B}^{\mathrm{T}}) & \mathbf{Q}^{-1}\mathbf{C} \\ -\mathbf{I} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix}_{,2} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}, \tag{10}$$

being the matrix $\mathbf{Q} = \mathbf{A} - \rho v^2\, \mathbf{D}$ non singular.

Let us find the spectrum and corresponding eigenvectors of the 8×8 matrix of coefficients in equation (10), namely the values of ω_k, $\mathbf{e}^k$ and $\mathbf{f}^k$ satisfying the following eigenvalue problem

$$\begin{bmatrix} \mathbf{Q}^{-1}(\mathbf{B}+\mathbf{B}^{\mathrm{T}}) - \omega_k\, \mathbf{I} & \mathbf{Q}^{-1}\mathbf{C} \\ -\mathbf{I} & -\omega_k\, \mathbf{I} \end{bmatrix} \begin{pmatrix} \mathbf{e}^k \\ \mathbf{f}^k \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}, \tag{11}$$

or equivalently

$$[\mathbf{C} - \omega_k (\mathbf{B} + \mathbf{B}^{\mathrm{T}}) + \omega_k^{\,2}\, \mathbf{Q}]\, \mathbf{f}^k = \mathbf{0}, \qquad \mathbf{e}^k = -\,\omega_k\, \mathbf{f}^k. \tag{12}$$

The eigenvalues ω_k are the roots of the characteristic equation

$$\det[\mathbf{C} - \omega_k (\mathbf{B} + \mathbf{B}^{\mathrm{T}}) + \omega_k^{\,2}\, \mathbf{Q}] = 0, \tag{13}$$

which admits the following purely imaginary roots with positive imaginary parts

$$\omega_1 = \frac{\mathrm{i}}{\sqrt{1-m_1^2}}, \qquad \omega_2 = \frac{\mathrm{i}}{\sqrt{1-m_2^2}}, \qquad \omega_3 = \mathrm{i}, \tag{14}$$

as well as the corresponding conjugate roots with negative imaginary part. In (14) we introduced the Mach numbers

$$m_1^2 = \frac{\rho v^2}{2\mu+\lambda-k_3/\delta}, \qquad m_2^2 = \frac{\rho v^2}{\mu-k_3/\delta}, \tag{15}$$

where $\delta = k_1/k_3$. Note that the algebraic multiplicity of the roots ω_3 is two, whereas the eigenvalues ω_1 and ω_2 are distinct. Moreover, for subsonic crack propagation both m_1 and m_2 are smaller than 1.

The eigenvectors ($\mathbf{e}^k$, $\mathbf{f}^k$) corresponding to each eigenvalue ω_k, for k = 1, 2, 3, are given by the non-trivial solution of the system (11), namely

$$\begin{aligned}
\mathbf{f}^1 &= (\mathrm{i}\,\delta\, m_1^2, -\delta\, m_1^2\sqrt{1-m_1^2}, -\mathrm{i}\,(4-3\,m_1^2), (4-m_1^2)\sqrt{1-m_1^2}), \\
\mathbf{f}^2 &= (\mathrm{i}\,\delta\, m_2^2\sqrt{1-m_2^2}, -\delta\, m_2^2, -\mathrm{i}\,(4-m_2^2)\sqrt{1-m_2^2}, 4-3\,m_2^2), \\
\mathbf{f}^3 &= (0, 0, -\mathrm{i}, 1),
\end{aligned} \tag{16}$$

and, thus, according to $(12)_2$:

$$\begin{aligned}
\mathbf{e}^1 &= \left(\frac{\delta\, m_1^2}{\sqrt{1-m_1^2}}, \mathrm{i}\,\delta\, m_1^2, -\frac{4-3\,m_1^2}{\sqrt{1-m_1^2}}, -\mathrm{i}\,(4-m_1^2)\right), \\
\mathbf{e}^2 &= \left(\delta\, m_2^2, \frac{\mathrm{i}\,\delta\, m_2^2}{\sqrt{1-m_2^2}}, -(4-m_2^2), -\mathrm{i}\,\frac{4-3\,m_2^2}{\sqrt{1-m_2^2}}\right), \\
\mathbf{e}^3 &= (0, 0, -1, -\mathrm{i}).
\end{aligned} \tag{17}$$

Note that the eigenvalue problem (11) is degenerate, since there exists only one eigenvector for each of the repeated eigenvalues ω_3 and $\overline{\omega}_3$ and, thus, their geometric multiplicity is less than the corresponding algebraic multiplicity. In this case, a generalized eigenvector ($\mathbf{e}^4$, $\mathbf{f}^4$), which is linearly independent of the previous three eigenvectors, can be defined for the repeated eigenvalue ω_3 from the solution of the following linear system [7]

$$\begin{bmatrix} \mathbf{Q}^{-1}(\mathbf{B}+\mathbf{B}^{\mathrm{T}})-\omega_3\,\mathbf{I} & \mathbf{Q}^{-1}\mathbf{C} \\ -\mathbf{I} & -\omega_3\,\mathbf{I} \end{bmatrix} \begin{pmatrix} \mathbf{e}^4 \\ \mathbf{f}^4 \end{pmatrix} = \begin{pmatrix} \mathbf{e}^3 \\ \mathbf{f}^3 \end{pmatrix}. \tag{18}$$

Therefore, the generalized eigenvectors $\mathbf{e}^4$ and $\mathbf{f}^4$ are given by the following equations

$$(\mathbf{B} + \mathbf{B}^{\mathrm{T}} - \omega_3\,\mathbf{Q})\,\mathbf{e}^4 + \mathbf{C}\,\mathbf{f}^4 = \mathbf{Q}\,\mathbf{e}^3, \qquad \mathbf{e}^4 = -\,\omega_3\,\mathbf{f}^4 - \mathbf{f}^3, \tag{19}$$

namely

$$\mathbf{f}^4 = \left(\frac{\delta}{4}\Big(2+\frac{\rho v^2}{\lambda+\mu}\Big), \mathrm{i}\,\frac{\delta}{4}\Big(2-\frac{\rho v^2}{\lambda+\mu}\Big), \frac{\delta(\rho v^2)^2}{8k_3(\lambda+\mu)}, 0\right), \tag{20}$$

$$\mathbf{e}^4 = \left(-\mathrm{i}\,\frac{\delta}{4}\Big(2+\frac{\rho v^2}{\lambda+\mu}\Big), \frac{\delta}{4}\Big(2-\frac{\rho v^2}{\lambda+\mu}\Big), \mathrm{i}\left[1-\frac{\delta(\rho v^2)^2}{8k_3(\lambda+\mu)}\right], -1\right). \tag{21}$$

Let us define the 4×4 matrices $\mathbf{E}$ and $\mathbf{F}$ such that their columns are the eigenvectors $\mathbf{e}^k$ and $\mathbf{f}^k$, respectively, for k = 1, 2, 3, 4, which have been defined in (16), (17), (20) and (21). Then, equations (11), for k = 1, 2, 3, and (18) and their complex conjugate relations can be written in the compact form

$$\begin{bmatrix} \mathbf{Q}^{-1}(\mathbf{B}+\mathbf{B}^{\mathrm{T}}) & \mathbf{Q}^{-1}\mathbf{C} \\ -\mathbf{I} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{E} & \overline{\mathbf{E}} \\ \mathbf{F} & \overline{\mathbf{F}} \end{bmatrix} = \begin{bmatrix} \mathbf{E} & \overline{\mathbf{E}} \\ \mathbf{F} & \overline{\mathbf{F}} \end{bmatrix} \begin{bmatrix} \mathbf{W}+\mathbf{N} & \mathbf{0} \\ \mathbf{0} & \overline{\mathbf{W}+\mathbf{N}} \end{bmatrix}, \tag{22}$$

where $\mathbf{W}$ and $\mathbf{N}$ are the following semisimple and nilpotent matrices

$$\mathbf{W} = \mathrm{diag}(\omega_1, \omega_2, \omega_3, \omega_3), \qquad \mathbf{N} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{23}$$

respectively, in agreement with the Jordan decomposition theorem of linear algebra. Let us define the vector $\mathbf{g}(x_1, x_2)$ such that

$$\begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} = \begin{bmatrix} \mathbf{E} & \overline{\mathbf{E}} \\ \mathbf{F} & \overline{\mathbf{F}} \end{bmatrix} \begin{pmatrix} \mathbf{g} \\ \overline{\mathbf{g}} \end{pmatrix} = 2\,\mathrm{Re} \begin{pmatrix} \mathbf{E}\,\mathbf{g} \\ \mathbf{F}\,\mathbf{g} \end{pmatrix}, \tag{24}$$

then, the introduction of (24) into the differential system (10), by using the relation (22) gives

$$\mathbf{g}_{,1} + (\mathbf{W}+\mathbf{N})\,\mathbf{g}_{,2} = \mathbf{0}, \tag{25}$$

and its complex conjugate relation. The differential system (25) writes explicitly:

$$g_{k,1} + \omega_k\, g_{k,2} = 0, \quad \text{for } k = 1, 2, 4, \qquad g_{3,1} + \omega_3\, g_{3,2} = -\, g_{4,2}. \tag{26}$$

Hence, by introducing the complex variables

$$z_k = x_1 + \frac{x_2}{\omega_k} = x_1 + \mathrm{i}\, x_2 \sqrt{1 - m_k^2}\,, \qquad \text{for } k = 1, 2, 3, 4, \tag{27}$$

with $m_3 = m_4 = 0$, so that $z_3 = z_4 = x_1 + \mathrm{i}\, x_2 = z$, the systems (26) reduce to

$$\frac{\partial g_k}{\partial \bar{z}_k} = 0, \quad \text{for } k = 1, 2, 4, \qquad \frac{\partial g_3}{\partial \bar{z}_3} = -\, g_{4,2}. \tag{28}$$

Therefore, the complex function $g_k(x_1, x_2)$, for $k = 1, 2, 4$, must be an analytic function of the complex variable z_k, namely $g_k = g_k(z_k)$, whereas the function $g_3(x_1, x_2)$ must satisfy the condition:

$$\frac{\partial g_3}{\partial \bar{z}} = -\frac{d g_4}{d z}\frac{\partial z}{\partial x_2} = -\,\mathrm{i}\; g_4'(z), \tag{29}$$

and thus by direct integration from (29) it follows that

$$g_3(x_1, x_2) = h_3(z) - \mathrm{i}\; \bar{z}\; g_4'(z), \tag{30}$$

where $h_3(z)$ is an arbitrary analytic function of the complex variable z.

The displacement derivatives and stress fields, collected in the vectors **a**, **b**, **p** and **q**, in term of the analytic functions $g_1(z_1)$, $g_2(z_2)$, $g_4(z)$ and $h_3(z)$, follow from (24) and (7) as

$$\mathbf{a} = 2\,\mathrm{Re}[\mathbf{E}\,\mathbf{g}], \qquad \mathbf{b} = 2\,\mathrm{Re}[\mathbf{F}\,\mathbf{g}], \qquad \mathbf{p} = 2\,\mathrm{Re}[\mathbf{G}\,\mathbf{g}], \qquad \mathbf{q} = 2\,\mathrm{Re}[\mathbf{H}\,\mathbf{g}], \tag{31}$$

where **G** and **H** are the following matrices

$$\mathbf{G} = \mathbf{A}\,\mathbf{E} + \mathbf{B}\,\mathbf{F}, \qquad \mathbf{H} = \mathbf{B}^{\mathrm{T}}\,\mathbf{E} + \mathbf{C}\,\mathbf{F}. \tag{32}$$

The stress and displacement distribution will be known once the analytic functions $g_1(z_1)$, $g_2(z_2)$, $g_4(z)$ and $h_3(z)$ have been determined for the boundary conditions of the considered problem.

4. Semi-infinite crack loaded on the crack surfaces

A semi-infinite rectilinear crack steadily propagating in a QC solid is considered. An uniform distribution of shear and normal phonon stresses denoted with τ_0 and σ_0, respectively, is applied to a segment of length L on the crack surfaces. Moreover, phonon and phason stress fields are assumed to vanish at infinity, so that the generalized stress vectors (5) must vanish at infinity as well. Let us denote with $\mathbf{g}(z)$ the vector which collects the functions $g_k(z_k)$, for $k = 1, 2, 3, 4$. According to (30), this vector can be written as

$$\mathbf{g}(z) = \mathbf{h}(z) - \mathrm{i}\ \bar{z}\ \mathbf{N}\,\mathbf{h}'(z), \tag{33}$$

being $h_k(z_k) = g_k(z_k)$, for $k = 1, 2, 4$. The introduction of (33) in $(31)_{1,4}$ gives

$$\mathbf{a} = 2\,\mathrm{Re}[\mathbf{E}\,(\mathbf{h} - \mathrm{i}\ \bar{z}\ \mathbf{N}\,\mathbf{h}')], \qquad \mathbf{q} = 2\,\mathrm{Re}[\mathbf{H}\,(\mathbf{h} - \mathrm{i}\ \bar{z}\ \mathbf{N}\,\mathbf{h}')]. \tag{34}$$

Continuity of the phonon and phason tractions (4) along the x_1 axis requires

$$\mathbf{q}^{+}(x_1, 0) = \mathbf{q}^{-}(x_1, 0), \qquad \text{for} \quad -\infty < x_1 < \infty. \tag{35}$$

Continuity of the phonon and phason displacements along the positive x_1 axis ahead of the crack tip requires

$$\mathbf{a}^{+}(x_1, 0) = \mathbf{a}^{-}(x_1, 0), \qquad \text{for} \quad x_1 > 0. \tag{36}$$

The considered loading conditions on the crack surfaces imply:

$$\mathbf{q}(x_1, 0) = \begin{cases} \mathbf{q}_0 & \textit{for} \quad -L < x_1 < 0, \\ \mathbf{0} & \textit{for} \quad x_1 < -L, \end{cases} \tag{37}$$

where $\mathbf{q}_0 = (-\tau_0, -\sigma_0, 0, 0)$. From condition (35) and result $(34)_2$ it follows that the function

$$\mathbf{j}(z) = \mathbf{H}\,[\mathbf{h}(z) - \mathrm{i}\,z\,\mathbf{N}\,\mathbf{h}'(z)] - \overline{\mathbf{H}}\,[\overline{\mathbf{h}}\,(z) + \mathrm{i}\,z\,\mathbf{N}\ \overline{\mathbf{h}}'\,(z)], \tag{38}$$

must be analytic in the whole complex plane and, thus, by using Liouville's theorem, it must be constant. Since the vector $\mathbf{h}(z)$ must vanish at infinity, then the value of the constant must necessarily be zero, and thus

$$\mathbf{H}\,[\mathbf{h}(z) \ - \mathrm{i}\,z\,\mathbf{N}\,\mathbf{h}'(z)] = \overline{\mathbf{H}}\,[\overline{\mathbf{h}}\,(z) \ + \mathrm{i}\,z\,\mathbf{N}\ \overline{\mathbf{h}}'\,(z)]. \tag{39}$$

From the continuity of displacements ahead of the crack tip (36) and result $(34)_1$ it follows that

$$\mathbf{E}\,[\mathbf{h}^{+}(x_1) - \mathrm{i}\,x_1\,\mathbf{N}\,\mathbf{h}'^{+}(x_1)] - \overline{\mathbf{E}}\,[\overline{\mathbf{h}}^{+}(x_1) + \mathrm{i}\,x_1\,\mathbf{N}\ \overline{\mathbf{h}}'^{+}(x_1)] =$$
$$= \mathbf{E}\,[\mathbf{h}^{-}(x_1) - \mathrm{i}\,x_1\,\mathbf{N}\,\mathbf{h}'^{-}(x_1)] - \overline{\mathbf{E}}\,[\overline{\mathbf{h}}^{-}(x_1) + \mathrm{i}\,x_1\,\mathbf{N}\ \overline{\mathbf{h}}'^{-}(x_1)], \qquad \text{for} \quad x_1 > 0, \tag{40}$$

where $\mathbf{h}^{\pm}(x_1) = \lim_{x_2 \to 0^{\pm}} \mathbf{h}(x_1 + \mathrm{i}\, x_2)$, so that $\overline{\mathbf{h}(z)}^{\pm} = \overline{\mathbf{h}}^{\mp}(x_1)$. From (39) one can obtain

$$\overline{\mathbf{h}}^{\pm}(x_1) + \mathrm{i}\, x_1\, \mathbf{N}\, \mathbf{h}'^{\pm}(x_1) = \overline{\mathbf{H}}^{-1} \mathbf{H}\, [\mathbf{h}^{\pm}(x_1) - \mathrm{i}\, x_1\, \mathbf{N}\, \mathbf{h}'^{\pm}(x_1)]. \tag{41}$$

Since the matrix Re[i $\mathbf{EH}^{-1}$] is not singular, then the introduction of (41) in (40) yields

$$\overline{\mathbf{h}}^{+}(x_1) - \mathrm{i}\, x_1\, \mathbf{N}\, \mathbf{h}'^{+}(x_1) - [\mathbf{h}^{-}(x_1) - \mathrm{i}\, x_1\, \mathbf{N}\, \mathbf{h}'^{-}(x_1)] = 0, \qquad \text{for} \quad x_1 > 0. \tag{42}$$

The loading conditions on the crack surfaces (37) and $(34)_2$ imply that

$$\mathbf{H}\, [\mathbf{h}^{+}(x_1) - \mathrm{i}\, x_1\, \mathbf{N}\, \mathbf{h}'^{+}(x_1)] + \overline{\mathbf{H}}\, [\overline{\mathbf{h}}^{-}(x_1) + \mathrm{i}\, x_1\, \mathbf{N}\, \overline{\mathbf{h}}'^{-}(x_1)] = \begin{cases} \mathbf{q}_0 & \mathit{for} \quad -L < x_1 < 0, \\ \mathbf{0} & \mathit{for} \quad x_1 < -L. \end{cases} \tag{43}$$

Then, by using result (39), from (43) one obtains

$$\mathbf{h}^{+}(x_1) - \mathrm{i}\, x_1\, \mathbf{N}\, \mathbf{h}'^{+}(x_1) + \mathbf{h}^{-}(x_1) - \mathrm{i}\, x_1\, \mathbf{N}\, \mathbf{h}'^{-}(x_1) = \begin{cases} \mathbf{H}^{-1}\mathbf{q}_0 & \mathit{for} \quad -L < x_1 < 0, \\ \mathbf{0} & \mathit{for} \quad x_1 < -L. \end{cases} \tag{44}$$

Conditions (42) and (44) define an inhomogeneous Riemann-Hilbert problem for the analytic vector function $\mathbf{h}(z) - \mathrm{i}\, z\, \mathbf{N}\, \mathbf{h}'(z)$, which admits the following solution vanishing at infinity [9]:

$$\mathbf{h}(z) - \mathrm{i}\, z\, \mathbf{N}\, \mathbf{h}'(z) = \frac{1}{\pi} \left\langle\!\left\langle \frac{1}{2\mathrm{i}} \log \frac{\sqrt{z_k} + \mathrm{i}\sqrt{L}}{\sqrt{z_k} - \mathrm{i}\sqrt{L}} - \sqrt{\frac{L}{z_k}} \right\rangle\!\right\rangle \mathbf{H}^{-1}\, \mathbf{q}_0. \tag{45}$$

Since the matrix $\mathbf{N}$ is nilpotent, and thus $\mathbf{N}^2 = \mathbf{0}$, then from (45) one obtains

$$\mathbf{N}\, \mathbf{h}'(z) = \frac{1}{\pi} \frac{L}{z+L} \sqrt{\frac{L}{z}}\, \mathbf{N}\, \mathbf{H}^{-1}\, \mathbf{q}_0. \tag{46}$$

The introduction of (45) and (46) in (33) then yields

$$\mathbf{g}(z) = \frac{1}{\pi} \left[\left\langle\!\left\langle \frac{1}{2\mathrm{i}} \log \frac{\sqrt{z_k} + \mathrm{i}\sqrt{L}}{\sqrt{z_k} - \mathrm{i}\sqrt{L}} - \sqrt{\frac{L}{z_k}} \right\rangle\!\right\rangle - \frac{2\, x_2}{z+L} \left(\frac{L}{z} \right)^{3/2} \mathbf{N} \right] \mathbf{H}^{-1}\, \mathbf{q}_0\,. \tag{47}$$

The phonon and phason displacements, collected in the vector $\mathbf{d} = (u_1, u_2, w_1, w_2)$, can be obtained by direct integration with respect to x_1 of the vector $\mathbf{a}$ in $(34)_1$, namely

$$\mathbf{d} = \frac{2}{\pi} \mathrm{Re}\{\mathbf{E} \left[\left\langle\!\left\langle \frac{z_k + L}{2\mathrm{i}} \log \frac{\sqrt{z_k} + \mathrm{i}\sqrt{L}}{\sqrt{z_k} - \mathrm{i}\sqrt{L}} - \sqrt{L\, z_k} \right\rangle\!\right\rangle + 4\, x_2 \left(\frac{1}{2\mathrm{i}} \log \frac{\sqrt{z} + \mathrm{i}\sqrt{L}}{\sqrt{z} - \mathrm{i}\sqrt{L}} + \sqrt{\frac{z}{L}} \right) \mathbf{N} \right] \mathbf{H}^{-1} \}\mathbf{q}_0. \tag{48}$$

The energy release rate G for a crack propagating in a QC can be obtained by generalizing the result found for linear elastic fracture mechanics displaying square root stress singularity [9]. By using (34), (45), and (46) the energy release rate becomes

$$G = -\frac{\pi}{2} \{ \lim_{r \to 0^+} \sqrt{r}\, \mathbf{q}(r) \} \cdot \{ \lim_{r \to 0^+} \sqrt{r}\, [\mathbf{a}(r\, \mathrm{e}^{\mathrm{i}\,\pi}) - \mathbf{a}(r\, \mathrm{e}^{-\mathrm{i}\,\pi})] \} = \frac{4L}{\pi}\, \mathbf{q}_0 \cdot \mathrm{Re}[\mathrm{i}\, \mathbf{E}\, \mathbf{H}^{-1}]\, \mathbf{q}_0. \tag{49}$$

5. Results

The results show that the distribution of the phonon stress field is similar to the corresponding results in linear elasticity fracture mechanics. The phason stress field, which arises from the coupling relationship between the phonon and the phason fields, also exhibits the square root singularity near the crack tip. The contours of phonon and phason stress components σ_{12}, σ_{22}, S_{12} and S_{22} normalized by σ_0, under Mode I loading conditions, namely for $\tau_0 = 0$, for $k_3 = 0.1\ k_1$ and $v^2 = 0.6\ \mu/\rho$ are plotted in Fig. 1, where the Cartesian coordinates are normalized by L. It is noted that a significant phason stress field is induced near the crack tip as a consequence of the coupling effect provided by the constitutive equations, also for a small value of the coupling parameter k_3.

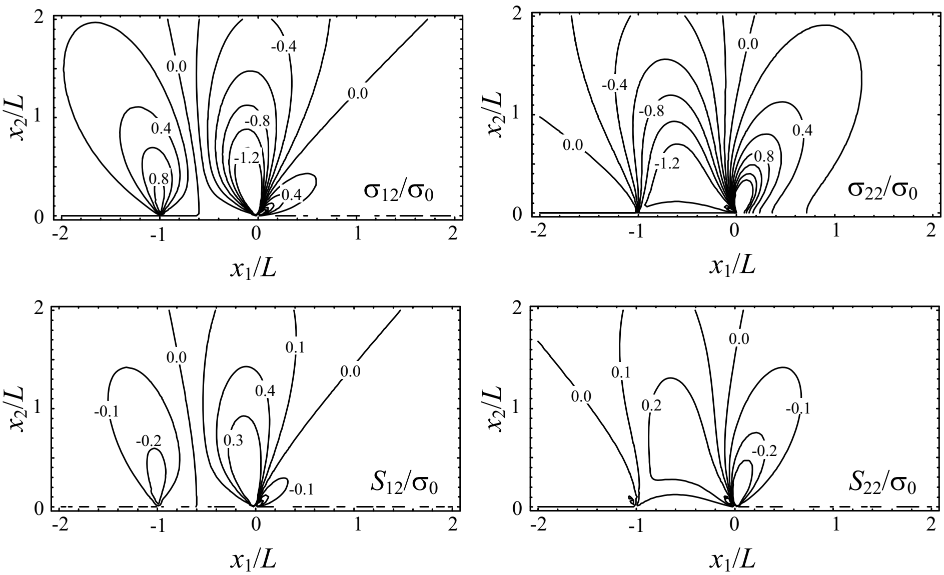

Figure 1. Contours of normalized phonon and phason stress components σ_{12}, σ_{22}, S_{12} and S_{22}, for $\tau_0 = 0$ (Mode I), $k_3 = 0.1\ k_1$ and $v^2 = 0.6\ \mu/\rho$.

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