A Probabilistic Model to Simulate Slender Reinforced Concrete Columns using Quasi-Monte Carlo Methods within SSJ Technology

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Keywords: reliability analysis, Quasi-Monte Carlo Method, Monte Carlo Method, low-discrepancy sequences, high-dimensional problems, Sobol´s sequence, SSJ, concrete columns

Abstract

The design procedure for slender reinforced concrete columns under uni-axial bent has been well established by Model Code 1990 [8]. The behaviour of bi-axially bent columns is a much more complex problem, particularly if buckling or stability problems occur. There are more than fifteen parameters in the numerical simulation, which constitute the system failure.

The random based Monte Carlo method and derived sampling methods are useful techniques to perform probabilistic analysis and to calculate probability of failure. If the observed problem is rather complex (slender columns) and the simulation of one single trail takes a lot of time, these methods are not useful. This paper discusses other types of simulation, which are known as Quasi-Monte Carlo (QMC) or low-discrepancy methods used to simulate slender columns. QMC methods have the potential to increase convergence rate of the probabilistic model. The underlying concept is to simulate the material and system parameters with quasi-randomized behaviour under given distributions. Therefore highly uniform point sets in unit hypercube are used. Stochastic Simulation in Java technology (SSJ) provides the necessary methods for implementation.

The finite element model bases on 3D nonlinear finite beam elements including nonlinear stress-strain-relationships, nonlinear geometry effects and any given boundary conditions.

Introduction

For safety design it is necessary to calculate a reliability index or to use a calibrated design code. Many practical design problems are so complex, that traditional reliability analysis using first order reliability method (FORM) or second order reliability method (SORM) fails to solve in common practice [1]. Consequently, in last years probabilistic methods that provide approximate solutions for such problems including implicit performance functions (see: Fig. 1) have been developed. Mostly random or stratified samplings have dominated the simulation technology for these structural problems. For example, Latin Hypercube sampling (LHS) method is used in reliability assessment of uni-axially bent concrete columns [5].

In this paper the Quasi-Monte Carlo method to structural problems including implicit performance function is adopted, to combine the advantages of random sampling with stratified sampling technology. Quasi-Monte Carlo Method became popular in finance simulation, especially in calculation of high-dimensional financial problems [2,3,7]. Quasi-Monte Carlo is a method of numerical integration that operates in the same way as Monte Carlo integration, but instead uses sequences of quasi-random numbers to compute the integral and reduce the variance. Quasi-random numbers are generated deterministically by low-discrepancy (LD) sequences.

The paper starts by formulating the problem in section 2. Then a brief summary of structural behaviour of slender concrete columns in section 3 is provided and the Quasi Monte Carlo method in chapter 4 is shown. Finally, in section 5, two numerical examples are given to demonstrate the validity of the method for slender reinforced concrete columns.

Problem Formulation

In fact, the fully non-linear problem of slender concrete columns is analytically unsolvable. Simulations of non-linear structural problems base on numerical models, for example finite element method (FEM). The fundamental differential equation will be solved iterative and incremental. In order to receive structural solution (safe or fail), a fast parameterized structural finite element model is necessary to run probabilistic simulation.

Ultimate limit states are related to the loss of load bearing capacity. The load bearing capacity of slender reinforced concrete columns is influenced by nonlinear material properties and stability behaviour (buckling) [6,11]. Table 1 summarized the main failure modes concerning slender concrete columns.

Tab. 1: Modes of failure

modes of failure	failure
cross-section failure	• *crushing of concrete* • *reinforcing steel, bond behaviour* • *shear failure of cross-section*
system failure	• *stability failure, buckling*

A performance function (or limit state function) has to be defined for limit state design including all failure modes. Mathematically this can be expressed in term of opposing the capacity R and the demand E as

$$g(x_1, x_2) = x_1 - x_2$$
$$g(R, E) = R - E. \tag{1}$$

If the capacity R is influenced by the demand E, the performance function becomes non-linear (see Fig. 1). Additionally for high-dimensional problems the performance function becomes implicit [] in n-dimensional space. The concerning equation can be defined as

$$g(x_1, x_2, \ldots\ldots, x_n) = g(X_n) < 0 \quad \text{or failure}. \tag{2}$$

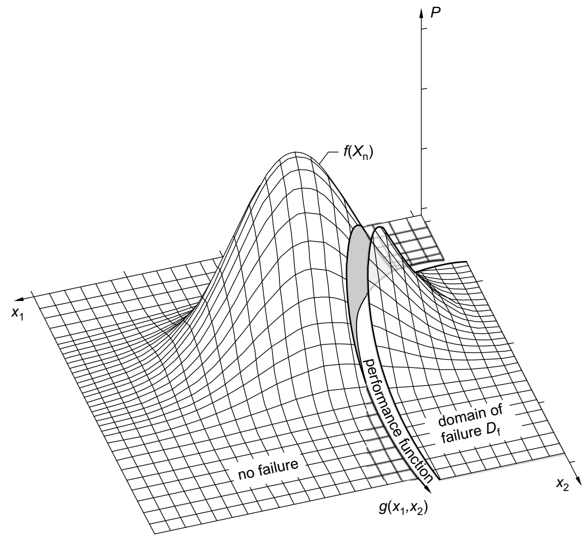

Fig. 1. Non-linear performance function in two dimensions

In general, the performance function can be a function with many variables, for example: load, material parameters, cross-section information, system boundaries, dimensions, slenderness and time depending behaviour. The parameters are subjected to uncertainties. For a fully probabilistic approach these parameters are described by a probability distribution. The kind of distribution depends on the physical behaviour of the parameter itself. The parameter distributions are shown on JCSS [12] probability code. Geometry related parameters as dimension of cross-section, and location and area of steel are defined deterministically in this research.

As the best reliability index the probability of failure will be defined. The integral of probability function on the domain of failure determines the probability of failure

$$P_{\mathrm{f}} = P(g(x_1, x_2, ..., x_{\mathrm{n}}) = \mathit{failure}) = \int_{D_f} f(x_1, x_2, ..., x_{\mathrm{n}}) dX \,. \tag{3}$$

The calculation can be seen as the solution of a multidimensional integral [2]. Because of very low target probabilities (10^{-3} – 10^{-6}) [12] accepted in structural design, there are high demands on integration method. The quadrature rules (within numerical evaluation) are designed all to compute one-dimensional integrals. To compute integrals in multiple dimensions, one approach is to use Monte Carlo method or derived sampling technologies. In simplest case, that means sampling randomly the entire unit hypercube (n-dimensions), mapping the parameters and calculate the structural problem for that set of variables. Various sampling technologies can be used:

- random sampling
- stratified sampling (LHS, updated LHS)
- quasi-random sampling within QMC method.

Within this research QMC method will be used for performing reliability analysis of slender columns. The generalised probabilistic model used for this analysis is shown in following scheme. The SSJ technology package [4] provides the basic functionality for implementing the probabilistic model.

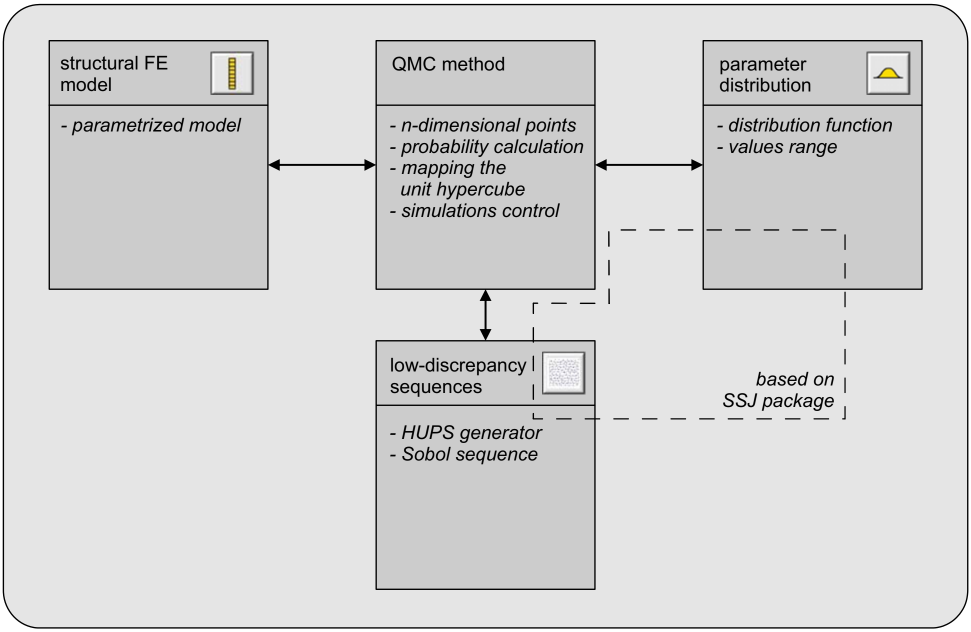

Fig. 2. Probabilistic model scheme

The whole model is implemented in Java programming language in an object oriented approach.

Structural Background

The parameterized numerical FE model is the basis for the general probabilistic approach. Failures in this model produce systematic errors in the reliability analysis. The realistic behaviour of slender columns must be taken into account. The theoretical assumptions of the structural FE model are:

- non-linear compressive stress-strain relation of concrete [8]
- non-linear stress-strain relation of reinforcing steel
- plane cross-section hypothesis
- 3D- beam elements with two nodes
- Newton Raphson system solver
- theory of large deflection
- beam elements.

The advantage of a beam element simulation is, that it can be run in a reasonable amount of computational time, to reduce the basic simulation time for probabilistic approach. The major disadvantage may be the ignored effects of stirrup-modelling. As seen in [6,10], the main reason to use stirrups is to prevent reinforcement bars from buckling and not to produce tri-axial stress states.

Quasi-Monte Carlo Method

Solving non-linear structural problems takes a lot of computational time. Because of that many variance reduction techniques have been developed, to solve the multi-dimensional integral (see Eq. 3). One promising approach of variance reduction is the use of low-discrepancy (LD) sequences together with quasi-random sampling [9]. An LD sequence generates integration point sets X_n, which are more evenly distributed over the unit hypercube than typical pseudo-random points.
The concept of low-discrepancy is associated with the property, that the successive numbers are added in a position farthest as possible from others points and avoiding point clustering. Following figure illustrates the *van der Corput* sequence [9].

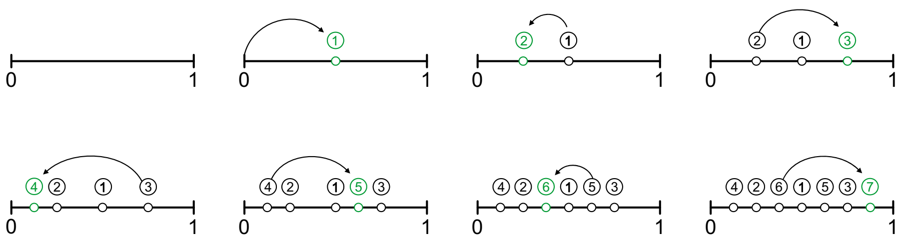

Fig. 3. *Van der Corput* sequence in one-dimension

Following main LD sequences for multi-dimensional analysis are providing by SSJ technology:

- *Sobol′* sequence [9]
- *Faure* sequence
- *Halton* sequence
- *Kobolov-Lattice* sequence.

The main reason to favour the use of quasi-random points is the reproducibility of the results in high accuracy. In contrast to LHS or other stratified sampling methods, the use of LD sequences guarantees the uniformity of samples when bounded simulation number extension occurs. Adding more simulation points is possible without rerunning the whole simulation. QMC method allows auto-stopping rules base on result observation. Within the presented research the *Sobol′* sequence is used for probabilistic simulation.
Low discrepancy methods have the potential to increase convergence rate from $O(1/\sqrt{num})$ associated with MC method to nearly $O(1/num)$. The accurate convergence rate is depending on the problem. The results are better if the integrand (see Eq. 3) is smooth [7]. QMC approximates the integral using

$$\int_{D_f} f(x_1, x_2, \ldots\ldots, x_n)dX = \frac{1}{num} \cdot \sum_{i=1}^{num} f(x_1, x_2, \ldots\ldots, x_n) \tag{4}$$

highly uniform points in the unit hypercube.

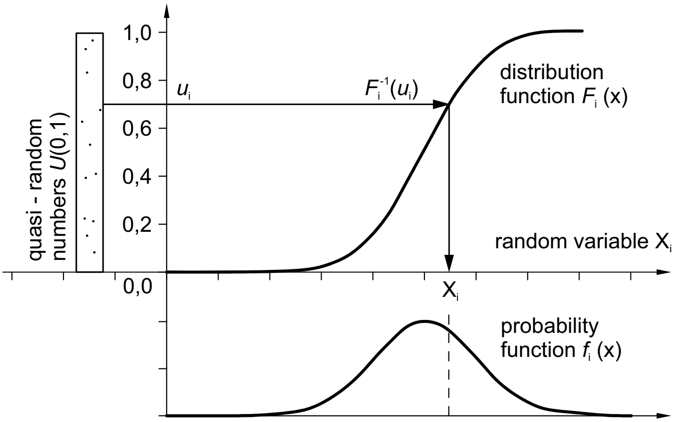

Fig. 4. Computation of the random variable

The nonlinear structural FE model performs simulation with QMC generated point sets of the input parameters. Figure 4 describes the mapping process using inverse function.

Numerical Examples

The numerical behaviour of reliability analysis of slender reinforced concrete columns is treated in the following two test examples. The only parameters varied in the tests are the column length and the load. That fact allows studying the effect of slenderness. Figure 5 and Table 3 shows the different parameters and the boundary conditions of the observed problems. To validate the results of the model, the problem will be simplified in that way, that only two parameters are modelled with uncertainties. The simplified problem has two dimensions in probability space. Other parameters considered in this whole structural problem, such as eccentricity, Youngs´ modulus, reinforcement ratio, columns geometry or reinforcement material have also a significant influence on the reliability calculation. For comparison reasons these parameters will be ignored within these research.

Tab. 3: Probability parameters and distributions

parameter	distribution type	mean value	2nd parameter
concrete strength	*log-normal*	$f_{cm} = 34N/mm^2$	$\sigma = 5N/mm^2$
load effects	*normal*	a) $\mu_m = 100kN$ b) $\mu_m = 200kN$	*CV = 10%*
other	*deterministic*		

The slender columns were hinged at the ends and the load was applied with an eccentricity of 20 mm in y- and z- direction. The ultimate loads are 153,9 kN of column a) and 271,8 kN of column b). It is clear that the column capacity is strongly affected by the total length (implicit slenderness).

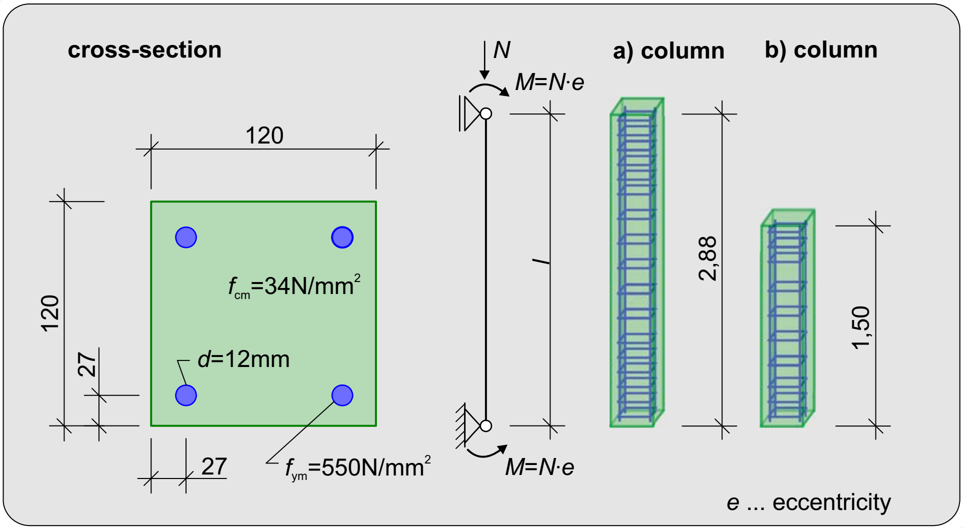

Fig. 5. Characteristics of discussed examples

The two explained numerical examples of bi-axially bent concrete columns are simulated with QMC and MC integration. SSJ provides both methods. The probabilities of failure were calculated using Eq. 4 for the considered problems. Some of the obtained results are presented in the following diagrams (see Fig. 6). Considerable differences in convergence can be seen between QMC and Crude MC Method. Due to the low dimension of the simplified problem approximately 5000 simulations are enough to calculate adequate results.

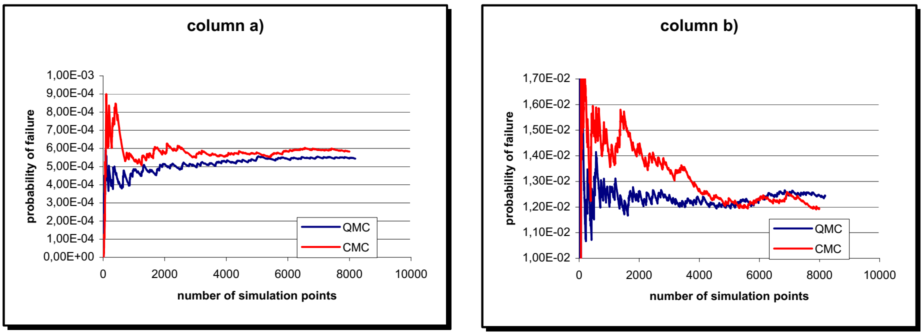

Fig. 6. Results of the calculation examples

It is observed that the QMC produced probability function is relative smooth in both cases. In the numerical illustration, the probabilities of failures are $p_f = 5.5 \cdot 10^{-4}$ and $p_f = 1.24 \cdot 10^{-2}$.

Conclusion

As a result of the presented research, a new technique for variance reduction for structural problems is proposed. The results of the two performed tests and the theoretical background of Quasi-Monte Carlo method presented here allow the following conclusions. The QMC method can be used in probabilistic simulation for structural problems with implicit performance functions. In comparison with traditional MC simulation, the proposed technique shows better performance in a way that the whole number of simulations can be reduced. In addition to produce smaller errors, the QMC method holds another big advantage. The relative smoothness of probability result-function favours the method for sensitivity analysis. SSJ technology provides the necessary functionality for quasi-random numbers.

References

[1] Rubinstein, R.Y.: *Simulation and the Monte Carlo Methods*, Wiley, New York (1981)

[2] P. Glassermann: *Monte Carlo Methods in Financial Engineering* Springer-Verlag NY Inc., New York (2004)

[3] G. Lacher: *Digital point sets: analysis and application*, pp. 167-222 in: Random and Quasi-Random Point Sets, Springer-Verlag Berlin (1998)

[4] Information on http://www.iro.umontreal.ca/~simardr/ssj/indexe.html

[5] Strauss, A., Bergmeister, K., März, S.: *Zuverlässigkeitsbetrachtungen exzentrisch belasteter Stahlbetonstützen*. Beton- und Stahlbetonbau 102, Heft 4 (2007)

[6] Claeson, C., Gylltoft, K.: *Slender High-Strength Concrete Columns Subjected to Eccentric Loading*, Journal of Structural Engineering (1998), pp.233-240

[7] Galanti, S. & Jung, A.: *Low-Discrepancy Sequences - Monte Carlo Simulation of Option Prices*. Journal of Derivatives (1997), pp.63-83

[8] MC 90, CEB-FIP Model Code 1990. *Comite Euro-International Du Beton*. Thomas Telford Services Ltd., London (1991)

[9] Niederreiter, H.: *Random Number Generation and Quasi-Monte Carlo Methods*, Society for Industrial and Applied Mathematics, Philadelphia (1992)

[10] Gaughan, J.: *Computer analysis of slender concrete columns*. Journal of the Structural Division, Proceedings of the American Society of Civil Engineers, No. 10894, ASCE (1974), pp. 2123-2141

[11] Szerszen, M., Szwed, A., Nowak, A.S.: *Reliability Analysis for Eccentrically Loaded Columns*. Technical Paper, ACI Structural Journal, Title 102 (2005), pp. 676 – 688

[12] JCSS(2000), *Probabilistic Model Code. Joint Comitee on Structural Safety*. http://www.jcss.ethz.ch (2008)

[13] Leemis, L.M.: *Reliability – Probabilistic Models and Statistical Methods* , Prentice-Hall, Englewood Cliffs, New Jersey 07632 (1995)