A Plasticity-like Framework for Fracture Mechanics

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Abstract. An incremental description of (linear elastic) fracture mechanics is presented which shows a perfect analogy with plasticity theory. The formulation of a generic criterion stemming from the associated plasticity theory is presented and its feature discussed. The analogy between plasticity and quasi-static crack growth leads also to a new algorithm for crack propagation for an arbitrary number of cracks in multi-connected materials, which is driven by the increment of external actions. Stability of crack path under mode I loading is finally analyzed.

Introduction

Fracturing process reveals three distinct phases [1]: loading without crack growth, stable crack growth an unstable crack growth. During crack advancing, energy dissipation takes place in the process-region, in the plastic region outside the process region, and eventually in the wake of plastic region. When the fracture process is idealized to infinitesimally small scale yielding, energy dissipation during crack growth is concentrated at the crack tip. This assumption together with linear elasticity is assumed in the present note, making use of Hooke's law without limitation of stress and strain magnitudes: the stress-strain fields in the crack tip vicinity is uniquely determined by the stress intensity factors (SIFs).

Similarly to the determination of the "elastic limit", the concept of incipient crack growth is difficult to identify: in both cases, the difficulty is solved by a convention. Onset of crack growth is governed theoretically by a local condition, describing when the process region reaches a critical state which, in most cases of engineering interest, is independent on body and loading geometry: this property is termed autonomy [2]. Several criteria, the Maximum Tensile Stress [3], the Maximum Shear Stress [4], the apparent Crack Extension Force [5], and the Strain Energy Density [6] to cite but a few, stem from the crack configuration "at the onset of propagation": they have been extensively represented in the SIFs plane $K_1 - K_2$. Many other criteria are grounded on the stress and strain fields in the "propagated configuration" as the crack elongation approaches zero from above, among them the Local Symmetry [7] and the Maximum Energy Release Rate [8, 9] criterion. It is natural therefore to analyze these criteria into a different plane, that in the rest of the paper will be named the Amestoy-Leblond plane.

Even if the total amount of stable crack growth does not obey the property of autonomy, being dependent on the plastic region about the crack tip, stable crack growth is ruled by local conditions at the process region. The onset of unstable crack growth is, on the contrary, a result of a global instability. Analogously to plasticity [10], the global quasi-static fracture propagation problem consists in seeking an expression of the crack propagation rate for all three phases of the fracturing process. The question can be posed in the following way: given the state of stress and the history of crack propagation (if any), express the crack propagation rate (if any) as a function of the stress and of the history. Indeed this path of reasoning seems quite natural: though, most of algorithm for crack propagation are designed in the opposite way: they express the external load history as a function of the crack propagation rate [11, 12]. Whereas this approach is quite easy, it is not optimal in evaluating the critical point of the equilibrium path and further it seems to be unsuitable in the presence of many propagating cracks in multi-connected bodies.

For linear elastic fracture mechanics, the crack propagation problem is studied in section exploiting its analogy with plasticity theory. A maximum principle is stated, that expresses the maximum dissipation at the crack tip during propagation; from it, *associated flow rule and a propagation criteria for angle determination descend.* As an important implication, crack propagation angles from any aforementioned criteria can be eventually recovered *keeping the convexity of the safe equilibrium domain, the constraint $K_1^* \leq K_1^C$, the correct energy dissipation at the crack tip.* Consistency conditions lead to the formulation of an algorithm for crack advancing in section , which is *driven by the increment of external actions* (under the simplifying assumption of proportional loading) and *allows the evaluation of crack length increment and curvature at the crack tips of several cracks contemporarily advancing.* Stability of crack path under mode I loading, as it has been analyzed in [13], is recovered for slightly curved or kinked cracks and *extended to any crack propagation angle.*

Small strains and displacements hypothesis is assumed on a domain $\Omega = \bigcup_{n=1}^{N} \Omega_n \subset \mathbb{R}^2$, together with isotropic linear elastic constitutive law in all the N homogeneous closed domains $\bar{\Omega}_n$. Interfaces between domains are assumed to be rigid, i.e. relative displacements along each interface are not allowed. Loci $\Upsilon_i,\ i = 1, 2, ...,$ of possible displacement discontinuities $\mathbf{w}_i(\mathbf{x})$ are defined as usual - see [14] for details - inside of each domain Ω: the issues of interface cracks and of intersection between moving cracks and interfaces fall beyond the scopes of the present note.

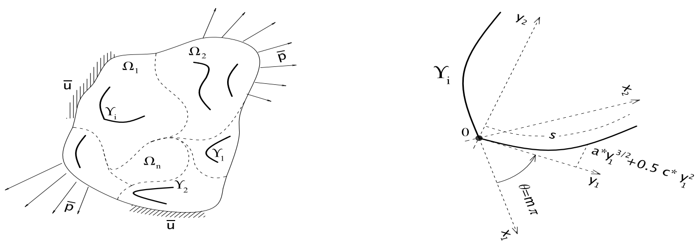

Figure 1: Notation. C denotes the curvature of the main branch at the crack tip, whereas a^* and C^* define the curvature of the elongated branch.

The structural response to the following quasi-static external actions is sought: tractions $\bar{\mathbf{p}}(\mathbf{x})$ on $\Gamma_p \subset \partial\Omega$, displacements $\bar{\mathbf{u}}(\mathbf{x})$ on $\Gamma_u \subset \partial\Omega$. They are all assumed to be *proportional*, i.e. that they vary only through multiplication by a time-dependent scalar $k(t)$, termed load factor, taken to be zero at initial time $t_0 = 0$ when the cracks attained their initial length. In the present note, "time" t represents any variable which monotonically increases in the physical time and merely orders events; the mechanical phenomena to study are time-independent. Domain forces are assumed to be zero.

The notation of [15], see also figure 1, will be used. In their celebrated paper [15], Amestoy and Leblond established the general form of the expansion of the stress intensity factors (SIFs) in powers of the crack extension length s, for a crack propagating in a two-dimensional body along an arbitrary kinked (by an angle $\theta = m\pi$) and curved path and evaluated the detailed form the functions of the geometric and mechanical parameters which appear in the expansion. Denoting with $\mathbf{K} = \{K_1, K_2\}$

the SIFs vector, the expansion of $\mathbf{K}$ at the extended crack tip in powers of s is of the general form:

$$\mathbf{K}(s) = \mathbf{K}^* + \mathbf{K}^{(1/2)}\sqrt{s} + \mathbf{K}^{(1)}\,s + O(s^{3/2}) \tag{1}$$

where $\mathbf{K}^*$, $\mathbf{K}^{(1/2)}$, $\mathbf{K}^{(1)}$ are given componentwise (using the Einstein summation convention) by

$$K_p^* = F_{pq}(m)\,K_q \tag{2}$$
$$K_p^{(1/2)} = G_p(m)\,T + a^* H_{pq}(m)\,K_q \tag{3}$$
$$K_p^{(1)} = Z_p + I_{pq}(m)\,b_q + C\,J_{pq}(m)\,K_q + a^* Q_p(m) T + a^{*2} L_{pq}(m)\,K_q + C^* M_{pq}(m)\,K_q \tag{4}$$

In these equations, T, and the b_qs are the non singular stress and coefficients of the $\sqrt{r}$ terms in the stress expansion at the original crack tip $\mathbf{0}$. The F_{pq}s, G_ps, H_{pq}s, I_{pq}s, J_{pq}s, Q_ps, L_{pq}s, and M_{pq}s are functions of the kink angle θ, which are termed universal because they obey to the autonomy concept[1]; finally, Z_p depends on the geometry of Ω.

A plasticity framework for LEFM

The definition of a "safe equilibrium domain" and of the "onset of crack propagation" as its closure remaind to the plasticity theory [10, 16]: they appear as the counterpart of the elastic domain and of the yield surface. Provided that merely the crack tip is considered as a material point, one is tempted to state that a crack tip is not going to propagate if the SIFs vector $\mathbf{K}^*$ belongs to the set:

$$\mathbb{E} = \left\{ \{K_1^*, K_2^*\} \in \mathbb{R}_0^+ \times \mathbb{R} \mid \varphi(K_1^*, K_2^*) < 0 \right\} \tag{5}$$

which is termed the "safe equilibrium domain". When $\mathbf{K}^* \in \mathbb{E}$ the material[2] surrounding the crack tip is experiencing a purely linear elastic behavior, eventually corresponding to an elastic unloading. The boundary of $\mathbb{E}$, $\partial\mathbb{E}$, is named the "onset of crack propagation surface":

$$\partial\mathbb{E} = \left\{ \{K_1^*, K_2^*\} \in \mathbb{R}_0^+ \times \mathbb{R} \mid \varphi(K_1^*, K_2^*) = 0 \right\} \tag{6}$$

and vectors $\mathbf{K}^* \notin \overline{\mathbb{E}}$ are ruled out. The definitions above implicity label the SIFs vector as an internal force for the LEFM problem, conjugated to a not yet specified internal variable.

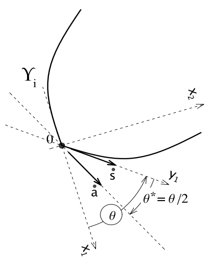

Figure 2: Definition of vector $\dot{\mathbf{a}}$. It is defined in the coordinate system $\{y_1, y_2\}$ assuming that angle θ^* is positive when counterclockwise, as usual. As a consequence, it always assumes the same absolute value and the opposite sign of θ.

[1]see [2] but also the excellent description in [1]
[2]in the linear elastic modelization, whose effectiveness is restricted to the small-scale yielding approach

At all material points experiencing plastic deformations, mechanical dissipation $\mathcal{D} > 0$ is induced; local dissipation inequality defines in plasticity (and more generally for standard dissipative systems) generalized strain rate as the conjugate to the generalized stress, as their product gives the rate of dissipation [10, 17]. In LEFM, mechanical dissipation is due to crack extension, for its irreversible nature [18]: it seems natural assuming as internal variable a quantity related to the quasi static crack tip velocity vector $\dot{\mathbf{s}}$, defined as the vector oriented with axis y_1 in figure 1 and with modulus equal to the quasi static velocity $\frac{ds}{dt}\big|_{s\to 0+}$ as the crack elongation s approaches zero from above. The internal variable is here termed "dissipation rate" vector $\dot{\mathbf{a}}$ and is defined as in figure 2: it is related to $\dot{\mathbf{s}}$ by the orientation defined through the kinking angle $\theta^* = \frac{1}{2}\,\theta$ and by its length, that will be proved to be equal to $\frac{ds}{dt}\big|_{s\to 0+}$ too.

A maximum principle - termed $\mathcal{D}$-principle - for LEFM is postulated as follows:

For given dissipation rate vector $\dot{\mathbf{a}}$ *among all possible SIFs on* $\mathbb{E}$*, the function*

$$\mathcal{D}(\mathbf{k}^*;\dot{\mathbf{a}}) = \mathbf{k}^* \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \dot{\mathbf{a}} \tag{7}$$

attains its maximum for the actual SIF vector $\mathbf{K}^*$*:*

$$\mathcal{D}(\mathbf{K}^*;\dot{\mathbf{a}}) = \max_{\mathbf{k}^*\in\mathbb{E}} \mathcal{D}(\mathbf{k}^*;\dot{\mathbf{a}}) \tag{8}$$

$\mathcal{D}$-principle - analogously to maximum dissipation in plasticity [16] - implies: i) associative flow rule in the Amestoy-Leblond plane (normality law):

$$\dot{\mathbf{a}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \frac{\partial \varphi}{\partial \mathbf{K}^*} \dot{\lambda} \tag{9}$$

ii) loading/unloading conditions in Kuhn-Tucker complementarity form:

$$\dot{\lambda} \geq 0, \qquad \varphi \leq 0, \qquad \dot{\lambda}\,\varphi = 0 \tag{10}$$

iii) convexity of safe equilibrium domain $\mathbb{E}$. The last of conditions (10) expresses the fact that $\dot{\lambda}$ and φ are not simultaneously nonzero: crack extension (i.e. $\dot{\lambda} > 0$) is possible when $\varphi = 0$, while negative φ implies that $\dot{\lambda}$ must be zero, in which case the behavior is linear elastic. *Consistency condition* can be deducted from (10) as usual (see e.g. [10]); they read:

$$\text{When } \varphi = 0, \qquad \dot{\lambda} \geq 0, \qquad \dot{\varphi} \leq 0, \qquad \dot{\lambda}\,\dot{\varphi} = 0 \tag{11}$$

Vectors $\dot{\mathbf{a}}$ and $\dot{\mathbf{s}}$ materialize the kinking angle $\theta^* = \frac{1}{2}\,\theta$, that can be obtained from the normality law as:

$$-\frac{\partial \varphi}{\partial {K_1}^*} \tan\theta^* = \frac{\partial \varphi}{\partial {K_2}^*} \tag{12}$$

The minus sign in the normality law (as well as the matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ in the $\mathcal{D}$ function) are required because when $K_2 > 0$ the kinking angle $\theta < 0$, as already noticed in the previous section.

Consider as crack propagation criterion the Maximum Energy Release Rate in the form[3]:

$$\varphi(\mathbf{K}^*) = \frac{1}{2}\,\frac{1-\nu^2}{E}\left(||\mathbf{K}^*||^2 - {K_1^C}^2\right) \tag{13}$$

[3]The quadratic form (13) is not degree-one homogeneous. It can be shown that outcome (15) can be obtained by using a degree-one homogeneous form as well.

It holds:

$$\dot{\mathbf{a}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \frac{\partial \varphi}{\partial \mathbf{K}^*} \dot{\lambda} = \frac{1-\nu^2}{E} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{K}^* \dot{\lambda} \tag{14}$$

and

$$\mathcal{D}(\mathbf{K}^*; \dot{\mathbf{a}}) = \mathbf{K}^* \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \dot{\mathbf{a}} = \frac{1-\nu^2}{E} ||\mathbf{K}^*||^2 \dot{\lambda} \geq 0 \tag{15}$$

It can be therefore concluded that: *i.* function $\mathcal{D}$ equals the energy dissipation at the crack tip due to an infinitesimal crack propagation $\dot{\lambda} = \dot{s}$; consequently, $\mathcal{D}$-principle is the counterpart of the postulate of maximum plastic work; *ii.* $\dot{\lambda} = \dot{s}$ is the actual "quasi-static crack propagation velocity" and $\lambda = s$ will coincide with the total crack propagation, provided that $\lambda = s = 0$ at the beginning of the crack propagation history.

An algorithm for crack propagation

Consider a monotonic sequence of instants $t_0 = 0$, t_1, ..., t_n, $t_{n+1} = t_n + \Delta t$. Assume all variables be known at t_n: the increments of the unknown are sought for the given variation over Δt of the external actions, governed by the variation $\Delta k = k(t_{n+1}) - k(t_n)$ of the load factor. With the aim of readability, the index $_n$ will be omitted in this section when unnecessary, with the only exception of instant t_n.

Assume that $J \geq 1$ crack tips are on the onset of propagation at t_n , that is for $j = 1, 2, ..., J$ the load factor $k(t_n)$ is such that $\{K_1^j, K_2^j\} \in \partial \mathbb{E}^j$. Because all angles of propagation merely depends upon the ratio $\alpha^j = \frac{K_2^j}{K_1^j}$, they are assumed to be known at time t_n and termed θ^j. Consistency conditions (10) can be invoked for the $j-$th crack elongation:

$$\dot{\varphi}^j = \sum_{i=1}^{2} \frac{\partial \varphi^j}{\partial K_i^*} \left(\frac{\partial K_i^*}{\partial k} \dot{k}(t_n) + \sum_{h=1}^{J} \frac{\partial K_i^*}{\partial a^h} \dot{a}^h(t_n) \right) \Bigg|_{\varphi^j = 0} = 0 \qquad j = 1, 2..., J \tag{16}$$

The linear system of equations (16) relates the J crack elongation "velocities" $\dot{a}^h$ to the variation of the load factor $\dot{k}$ at time t_n, which drives the loading process and can be assumed to be given. In order to achieve a more effective algorithm, an arc-length procedure [19] can be set up instead of assuming a given load factor variation, with the arc-length eventually adapted by the curvature of the equilibrium path. In both cases, the constraint $\dot{a}^h > 0$ for all $h = 1, 2..., J$ avoids troubles in the choice of the sign of variation $\dot{k}$.

In equation (16): K_i^* is always referred to the j-th crack, beacause any crack propagation criteria at a crack tip is merely dependent on SIFs at the same crack tip: for the sake of clearness the apex ^{j} is omitted for K_i^*; $\frac{\partial \varphi^j}{\partial K_i^*}$ depends on the selected criteria; $\frac{\partial K_i^*}{\partial k}$ is trivial because for given crack lengths the global behavior is purely linear elastic: therefore

$$\frac{\partial K_i^*}{\partial k}\Bigg|_{\varphi^j = 0} = \frac{K_i^*}{k}\Bigg|_{\varphi^j = 0}$$

factors $\frac{\partial K_i^*}{\partial a^h}$ depend on the global elastic problem and, besides intrinsic difficulties related to expansion (1), their evaluation is yet an on going research topic [20]. A way to circumvent such a drawback is assuming an expansion for a^h with respect to k at time t_n in the following form:

$$a^h(t) - a^h(t_n) = \sum_{\omega=1}^{\Omega} Q_\omega^h(a^1(t_n), ..., a^J(t_n)) \, [\, k(t) - k(t_n) \,]^\omega \qquad t > t_n \tag{17}$$

in which the maximum number of terms Ω is suitably selected. Assuming for instance $\Omega = 1$ leads to:

$$\begin{aligned} \dot{\varphi}^j &= \dot{k}(t_n) \sum_{i=1}^{2} \frac{\partial \varphi^j}{\partial K_i^*} \left(\frac{K_i^*(t_n)}{k(t_n)} + \sum_{h \neq j} \frac{\partial K_i^*}{\partial a^h} \frac{\partial a^h}{\partial k}\bigg|_{\varphi^j=0} + \frac{\partial K_i^*}{\partial a^j} \frac{\partial a^j}{\partial k}\bigg|_{\varphi^j=0} \right) \\ &= \dot{k}(t_n) \sum_{i=1}^{2} \frac{\partial \varphi^j}{\partial K_i^*} \left(\frac{K_i^*(t_n)}{k(t_n)} + \left[\sum_{h \neq j} \frac{\partial K_i^*}{\partial a^h} Q_1^h + \frac{\partial K_i^*}{\partial a^j} Q_1^j \right]_{\varphi^j=0} \right) \\ &= 0 \qquad j = 1, 2..., J \end{aligned}$$

Taking into account of expansions (1) and (17), one gets:

$$\begin{aligned} \dot{\varphi}^j &= \dot{k}(t_n) \sum_{i=1}^{2} \frac{\partial \varphi^j}{\partial K_i^*} \left[\frac{K_i^*(t_n)}{k(t_n)} + \sum_{h \neq j} \frac{\partial K_i^*}{\partial a^h} Q_1^h + \left(\frac{K_i^{(1/2)}}{2\sqrt{a^j(t) - a^j(t_n)}} + K_i^{(1)} \right) Q_1^j \right]_{\varphi^j=0} \\ &= \dot{k}(t_n) \sum_{i=1}^{2} \frac{\partial \varphi^j}{\partial K_i^*} \left[\frac{K_i^*(t_n)}{k(t_n)} + \sum_{h \neq j} \frac{\partial K_i^*}{\partial a^h} Q_1^h + K_i^{(1/2)} \frac{\sqrt{Q_1^j}}{2\sqrt{k(t) - k(t_n)}} + K_i^{(1)} Q_1^j \right]_{\varphi^j=0} \\ &= 0 \qquad j = 1, 2..., J \end{aligned}$$

The expression above states that coefficient Q_1^j of the linear contribution in expansion (17) at j-th crack tip is non vanishing if and only if[4] $\mathbf{K}^{(1/2)^j} = 0$, in which case:

$$\sum_{h \neq j} \frac{\partial \varphi^j}{\partial \mathbf{K}^*} \cdot \frac{\partial \mathbf{K}^*}{\partial a^h} Q_1^h + \frac{\partial \varphi^j}{\partial \mathbf{K}^*} \cdot \mathbf{K}^{(1)} Q_1^j = -\frac{1}{k(t_n)} \frac{\partial \varphi^j}{\partial \mathbf{K}^*} \cdot \mathbf{K}^* \qquad j = 1, 2..., J \tag{18}$$

According to equation (18), Q_1^j depends also on Q_1^h. If $\frac{\partial \mathbf{K}^*}{\partial a^h}$ shows an $o(1)$ dependency on $k(t) - k(t_n)$ (at this moment this is still an open problem) it comes out:

$$Q_1^j = -\frac{1}{k(t_n)} \frac{\frac{\partial \varphi^j}{\partial \mathbf{K}^*} \cdot \mathbf{K}^*}{\frac{\partial \varphi^j}{\partial \mathbf{K}^*} \cdot \mathbf{K}^{(1)}} \qquad j = 1, 2..., J \tag{19}$$

Assuming $\Omega = 2$ and taking $\mathbf{K}^{(1/2)^j} \neq 0$ for all $j = 1, 2, ..., J$ leads to[5]:

$$\begin{aligned} \dot{\varphi}^j &= \dot{k}(t_n) \sum_{i=1}^{2} \frac{\partial \varphi^j}{\partial K_i^*} \left(\frac{K_i^*(t_n)}{k(t_n)} + \sum_{h \neq j} \frac{\partial K_i^*}{\partial a^h} \frac{\partial a^h}{\partial k}\bigg|_{\varphi^j=0} + \frac{\partial K_i^*}{\partial a^j} \frac{\partial a^j}{\partial k}\bigg|_{\varphi^j=0} \right) \\ &= \dot{k}(t_n) \sum_{i=1}^{2} \frac{\partial \varphi^j}{\partial K_i^*} \left(\frac{K_i^*(t_n)}{k(t_n)} + 2\,(k(t) - k(t_n)) \left[\sum_{h \neq j} \frac{\partial K_i^*}{\partial a^h} Q_2^h + \frac{\partial K_i^*}{\partial a^j} Q_2^j \right]_{\varphi^j=0} \right) \\ &= 0 \qquad j = 1, 2..., J \end{aligned}$$

Assuming further that $\frac{\partial K_i^*}{\partial a^h}$ shows no singular behavior when $h \neq j$ and taking into account of expan-

[4]This result corresponds to assumption (100) at page 492 in [15].

[5]The general case $\mathbf{K}^{(1/2)^j} \neq 0$ only for some j is a trivial extension of the following.

sions (1) and (17), one gets:

$$\begin{aligned}
\dot{\varphi}^j &= \dot{k}(t_n) \sum_{i=1}^{2} \frac{\partial \varphi^j}{\partial K_i^*} \left[\frac{K_i^*(t_n)}{k(t_n)} + 2 \sum_{h \neq j} \frac{\partial K_i^*}{\partial a^h} Q_2^h \, (k(t) - k(t_n)) + K_i^{(1/2)} \, Q_2^j \, \frac{k(t) - k(t_n)}{\sqrt{a^j(t) - a^j(t_n)}} \right]_{\varphi^j=0} \\
&= \dot{k}(t_n) \sum_{i=1}^{2} \frac{\partial \varphi^j}{\partial K_i^*} \left[\frac{K_i^*(t_n)}{k(t_n)} + 2 \sum_{h \neq j} \frac{\partial K_i^*}{\partial a^h} Q_2^h \, (k(t) - k(t_n)) + K_i^{(1/2)} \sqrt{Q_2^j} \right]_{\varphi^j=0} \\
&= 0 \qquad j = 1, 2..., J
\end{aligned}$$

By neglecting the higher order term $k(t) - k(t_n)$, it comes out:

$$\sqrt{Q_2^j} = -\frac{1}{k(t_n)} \frac{\frac{\partial \varphi^j}{\partial \mathbf{K}^*} \cdot \mathbf{K}^*}{\frac{\partial \varphi^j}{\partial \mathbf{K}^*} \cdot \mathbf{K}^{(1/2)}} \qquad j = 1, 2..., J \tag{20}$$

According to equation (20), Q_2^j depends merely on the load factor $k(t_n)$ and on quantities pertaining to the j-th crack tip: the presence of all other cracks is reflected by vectors $\mathbf{K}^*(t_n)$ and $\mathbf{K}^{(1/2)}$. As this last term depends on the T-stress, so does Q_2^j; if a straight elongation is considered, i.e. $a^* = 0$ in equation (3), Q_2^j can be evaluated from (20).

In the more general case, a^* can be evaluated exploiting the normality law (12). In view of expansion (1), equation (12) becomes:

$$-\left(K_1^* + K_1^{(1/2)} \sqrt{s} \right) \tan \theta^* = K_2^* + K_2^{(1/2)} \sqrt{s}$$

whence the $zero-$order outcome:

$$\tan \theta^* = -\frac{K_2^*}{K_1^*}$$

The $\frac{1}{2}-$order equation:

$$-K_1^{(1/2)} \tan \theta^* = K_2^{(1/2)}$$

allows the evaluation of a^* in view of identity (3)

$$a^* = -\frac{G_2(\theta) - G_1(\theta) \tan \theta^*}{H_{1q}(\theta) K_q \tan \theta^* - H_{2q}(\theta) K_q} T \tag{21}$$

using the Einstein summation notation.

Concluding remarks

Within the present note, a relation between the angle of propagation θ and the angle θ^* of the form $\theta = 2\theta^*$ has been assumed. Indeed, a certain arbitrary is nested in this choice: it is the same degree of freedom actually present in the choice of the crack propagation criteria. A more involved relation $\theta = \varsigma(\theta^*)$ could be proposed in order to recover the crack propagation angle θ predicted by any criteria: in this way, the degree of freedom in selecting a crack propagation criteria is transposed in the selection of mapping ς. As long as the remaining part of the plasticity analogy is kept, in particular the maximum energy release rate criterion for the safe equilibrium domain (13), the maximum principle (7) and its descending outcome (15), any selection of ς keeps the convexity of the safe equilibrium domain in the $K_1 - K_2$ as well as in the Amestoy-Leblon d planes, the respect of the constraint $K_1^* \leq K_1^C$ in mixed mode crack propagation, the correct energy dissipation at the crack tip.

The topic of the present note shows promising features and developments. Extending to linear elastic fracture mechanics the enormous amount of knowledge pursued in the last decades in plasticity is fascinating indeed, on several perspectives: theoretical, computational, educational.

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