A Numerical Analyse of Plane crack in Strain Gradient for Fracture Prediction in Brittle Material

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Abstract: In this work, a new model is developed to predict the fracture in brittle material from a geometrical weakness presenting an arbitrary stress concentration. The main idea is to combine the strain gradient elasticity with a cohesive model. The proposed model allows us to consider the different stress concentrations, even if they present some singularity. This model uses three material parameters which are σ_c the ultimate stress, G_c the critical energy release rate and l the characteristic length which represent material heterogeneities at a microscopic scale as in most of the strain gradient theories. We also present the potential extension of the theory to interpret the size effect. The influence of the default size (hole, inclusions) on the fracture behaviour is considered. The present model was implemented into a finite element code. A triangle finite element of 36 degrees of freedom is used for massive elements. We developed a cohesive element in order to describe the fracture procedure on the crack path. The cohesive force is calculated to predict the initiation process of the crack. We compare the numerical results with experimental data on PMMA specimens with circular holes of different sizes. It is shown that the present model predicts an inverse first power relation between the tensile strength and the size of the pre-existing crack. This prediction is in accordance with experimental evidence. It is demonstrated that the effect of the volumetric strain-gradient term corresponds to the shielding of the applied loads leading to crack stiffening; hence the present model is able to capture the commonly observed phenomenon of high-effective fracture energies in brittle materials. In contrast with conventional fracture theories unable to describe well this size dependence, the introduction of the strain gradient theory together with the cohesive model seems to be an appropriate approach to brittle material fracture.

1. INTRODUCTION

A nature understanding of the crack's development process is very important in the research of mechanicals fracture materials. Experimental research affirms that the heterogeneity has an important influence on the material fracture. In real structures, the failures are often initiated from a few geometrical weaknesses near which stress concentrations are formed. In brittle materials, the crack initiation often followed by unstable crack propagation. Thus leads to the final failure of the structure.

The classical linear elasticity has many limitations because it is not valid near the crack tip with the stress value become infinity. Barenblatt(1962) [2] introduced the foundation of the cohesive model research of material fracture. In this model, Barenblatt supposes a transition region exists in the tip of the crack. The forces of cohesion act on the crack's faces when an external loads. These forces reach zero when the distance between two faces of a crack surpass the critical value δ_c. The physical idea is that the density factor is zero at the limit point of the cohesive region. Singular stress in classical theory becomes non-singular with cohesive model.

Limitations of the classical model are due to the dominance of micro-structural effects and local interaction in the material instead of the large scale which is more closes the nature physical of the deformation. In order to over come this limitations, we use the long-rang atomic interactive forces at a macro scale into the continuum theory. The effect of long-range atomic interaction is expressed by a characteristic length scale into the continuum theory. This is enough to eliminate stress singularity at the crack tip. Many theories combine non-local elasticity and higher order continuum like the couple stress theory of Mindlin and other gradient elasticity theories which we present in the following one of these.

2. GRADIENT ELASTICITY THEORY

In strain-gradient theory of elasticity, the strain energy density function is assumed to be a function of not only the first gradients, but also of the second gradients of the displacement field.

$$w = w(\varepsilon_{ij}, \partial_k \varepsilon_{ij}) \tag{2.1}$$

Following the principle of virtual work, variation of the total potential energy internal work is equal to the variation of work caused by the external forces. The macroscopic strain coincides with the micro-deformation.

$$\delta w_I = \delta w_E \tag{2.2}$$

with

$$\delta w_I = \delta \int_V w dV = \int_V \left(\sigma_{ij}\delta\varepsilon_{ij} + \mu_{ijk}\delta(\partial_i \varepsilon_{jk})\right) dV \tag{2.3}$$

where

$$\sigma_{ij} = \frac{\partial w}{\partial \varepsilon_{ij}} = \sigma_{ji}, \quad \mu_{kji} = \frac{\partial w}{\partial(\partial_i \varepsilon_{jk})} = \mu_{kij} \tag{2.4}$$

The second-order tensor σ_{ij}, which is conjugate in energy to the macroscopic strain, is symmetric, whereas the third-order tensor μ_{ijk} , which conjugates to the strain-gradient is called the double stress and is symmetric with the last two indices $\mu_{ijk} = \mu_{kji}$. The first index denotes the plane on which doubles tress is acting, the second index denotes the direction of the lever arm and the third index denotes the direction of action.

The variation of strain energy density in a volume V of the body with arbitrary variation of ε_{ij} is

$$\begin{aligned}
\delta w &= \frac{\partial w}{\partial \varepsilon_{ij}}\delta\varepsilon_{ij} + \frac{\partial w}{\partial(\partial_i \varepsilon_{jk})}\delta(\partial_i \varepsilon_{jk}) = \sigma_{ji}\delta\varepsilon_{ij} + \mu_{kji}\delta(\partial_i \varepsilon_{jk}) \\
&= \sigma_{ij}\delta u_{i,j} + \mu_{kji}\delta u_{i,jk} \\
&= [(\sigma_{ji}\delta u_i)_{,j} - \sigma_{ji,j}\delta u_i] + [(\mu_{kji}\delta u_{i,j})_{,k} - \mu_{kji,k}\delta u_{i,j}] \\
&= [(\sigma_{ji}\delta u_i)_{,j} - \sigma_{ji,j}\delta u_i] + [(\mu_{kji}\delta u_{i,j})_{,k} - (\mu_{kji,k}\delta u_{i,j})_{,j} + \mu_{kji,kj}\delta u_i] \\
&= [(\sigma_{ji} - \mu_{kji,k})\delta u_i]_{,j} - (\sigma_{ji,j} - \mu_{kji,kj})\delta u_i + (\mu_{kji}\delta u_{i,j})_{,k}
\end{aligned} \tag{2.5}$$

We apply the divergence theorem to equation (2.8). This we have:

$$\delta w_I = \int_S n_j(\sigma_{ji} - \mu_{kji,k})\delta u_i dS - \int_V (\sigma_{ji,j} - \mu_{kji,kj})\delta u_i dV + \int_S n_k \mu_{kji}\delta u_{i,j} dS \tag{2.6}$$

The variation of work done by the external forces:

$$\delta w_E = \int_S t_i \delta u_i dS + \int_S \tau_i D\delta u_i dS \qquad (2.7)$$

where t_i and τ_i are traction and double traction.

The form proposed by Aifantis, which is a special case of Mindlin theory, is as follow:

$$w = \frac{1}{2}\lambda\varepsilon_{ii}\varepsilon_{jj} + \mu\varepsilon_{ii,k}\varepsilon_{jj,k} + \mu\varepsilon_{ij,k}\varepsilon_{ij,k}) \qquad (2.8)$$

With the conjugate equation (2.4) we obtain:

$$\sigma_{ij} = \frac{\partial W}{\partial \varepsilon_{ij}} = \lambda\varepsilon_{pp}\delta_{ij} + 2\mu\varepsilon_{ij} \qquad \mu_{kji} = \frac{\partial W}{\partial \varepsilon_{ij,k}} = l^2(\lambda\varepsilon_{pp}\delta_{ij} + 2\mu\varepsilon_{ij})_{,k} \qquad (2.15)$$

3. COHESIVE MODEL THEORY

The cohesive forces depend on the separation distance between the two crack lips. This zone develops progressively as the remote load increases until the separation energy provided by the remote loads overcomes the cohesive energy due to the cohesive forces. Once this critical state is achieved, the crack propagation is supposed to occur. In general, we can define a cohesive potential $\varphi([\delta])$, where $[\delta]$ is the displacement jump between two faces of the crack, such that:

$$T = \frac{\partial \varphi}{\partial [\delta]} \qquad (3.1)$$

We define a cohesive potential as follows:

$$\varphi = \begin{cases} 0 & \delta_e < 0 \\ a\delta_{max}\left\{\frac{1}{1+m}\left(\frac{\delta_e}{\delta_{max}}\right)^{m+1} - \frac{1}{1+n}\left(\frac{\delta_e}{\delta_{max}}\right)^{n+1}\right\} & \text{for} \quad 0 \le \delta_e \le \delta_{max} \\ a\delta_{max}\left\{\frac{1}{m+1} - \frac{1}{n+1}\right\} & \delta_{max} < \delta_e \end{cases} \qquad (3.2)$$

with δ_e defined as the effective displacement jump, as follows:

$$\delta_e = \delta + l_c^{\,3}\frac{\partial^3 u_n}{\partial n^3} \qquad (3.3)$$

where a, m, n, l_c and δ_{max} are materials constants.

The dual quantity related to the normal separation is the cohesive force T. With (2.24), we have for $0 \le \delta_e \le \delta_{max}$:

$$T = \frac{\partial \varphi}{\partial [\delta]} = a\left[\left(\frac{\delta_e}{\delta_{\max}}\right)^m - \left(\frac{\delta_e}{\delta_{\max}}\right)^n\right]\frac{\delta}{\delta_e} \qquad T_e = \frac{\partial \varphi}{\partial [\delta_e]} = a\left[\left(\frac{\delta_e}{\delta_{\max}}\right)^m - \left(\frac{\delta_e}{\delta_{\max}}\right)^n\right]$$

(3.4)

with T_e is the effective cohesive force

In order to determine the value of $\delta_{\max}$ and a, we suppose that the effective cohesive traction T_e is limited by the ultimate stress of the material σ_c, which is measured by uniform tension tests. We obtain:

$$a = \frac{\sigma_c}{\left[n^{\frac{1}{1-n}} - n^{\frac{n}{1-n}}\right]} \qquad \delta_{\max} \frac{G_c\left(n^{\frac{1}{1-n}} - n^{\frac{n}{1-n}}\right)}{\sigma_c\left(\frac{1}{2} - \frac{1}{n+1}\right)} \tag{3.5}$$

4. FINITE ELEMENT FORMULATION

Massive elements

The strain gradient elasticity is:

$$\bar{\sigma}_{ji} = \lambda\varepsilon_{pp}\delta_{ij} + 2\mu\varepsilon_{ij} - l^2(\lambda\varepsilon_{pp}\delta_{ij} + 2\mu\varepsilon_{ij})_{,kk} \tag{4.1}$$

We remark that the second derivative of strain is defined if the displacement function exists upon differentiating three times. Hence, the finite element has to assume the continuity in displacement, displacement gradient and strain gradient at the nodes and all along the elements edges. The finite element used in this work to interpolate displacement field is a higher-order triangular with quintic polynomial like the figure 1.

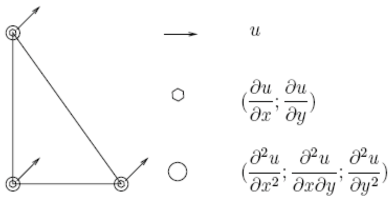

Figure 1: Massif element

It has three nodes with six degrees of freedom per node for each displacement component. There degrees of freedom are the displacement, its two derivatives and its three second derivatives. Hence, the total numbers of degrees of freedom per element are thirty six.

Cohesive elements

We define a 4-node cohesive element with nodes 1, 2 belonging to S_1 and the nodes 3,4 belonging to S_2 like the figure 2.

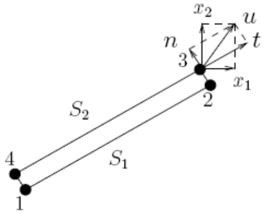

Figure 2: Cohesive element

The displacement jump between the two surfaces described in the global coordinate is therefore:

$$[\delta]=[u_x]n_x+[u_y]n_y=[u_{xy}]^T n \tag{4.7}$$

with

$$[u_x]=u_x{}^S-u_x{}^I \qquad [u_y]=u_y{}^S-u_y{}^I$$

where $[u_{xy}]=\{[u_x][u_y]\}^T$ is the displacement jump vector between the displacement vectors on S_1 and S_2 respectively.

5. RESULTS AND DISCUSSION

5.1. Verification of the model by the experimental results

This work is carried out on the shaped PMMA plates with a central hole under uniaxial tension. The mechanical characteristic of material are: the elastic modulus E=3000 MPa, the Poisson ratio $\nu=0.36$, the ultimate tensile stress $\sigma_c=72$ MPa, the critical release energy rate $G_c=290N/m$. The section of the specimens is 10x20mm. The diameters of central hole can be d = 0.25, 0.5, 0.75, 1, 1.5, 2, 3mm. Our numerical results are compared with the experimental results obtained by Li and Zhang [6].

In this work, we use the triangle element for strain elements and the cohesive elements are placed on the ligament in order to assess the fracture process. The remote load is applied on the top of the plate. The fracture loads are calculated by an incremental procedure. The displacement is applied gradually. At each step, iteration is carried out until convergence. When the optimal load is reached, the structure is completely separated at the ligament.

The size effect is observed by the test results using the different holes diameters. It is clear that the strength of the specimens depends strongly this diameter. The critical load increases as the holes diameters decreases. The unknown material parameter is the characteristic length l in the strain gradient element and l_c in the cohesive element. The results illustrated clearly the influence of l in the strength of the material. It is showed in the figure below. The experimental results are plotted to compare with of the numerical model. As a result, we see that the model describe well the size effect whatever the value of l. The resistant capacity of material increases with the increase of the value l. For PMMA we realize that with l=0.5mm the model corresponds to experimental results.

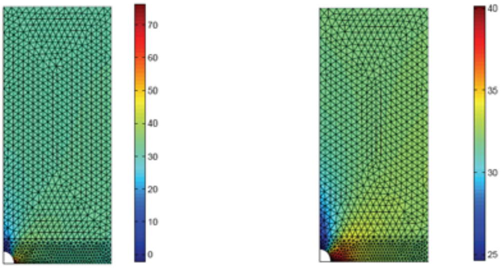

Figure 3: Local stress with L = 0, L = 0.5 and R = 0.5mm

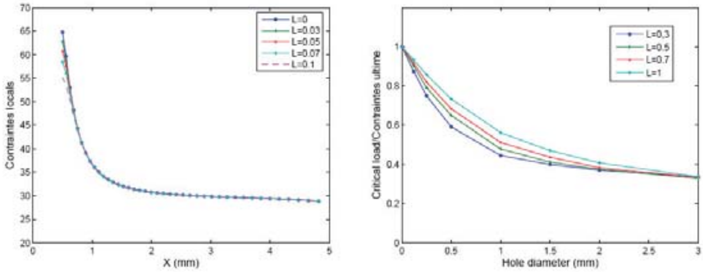

Figure 4: Predicted local stress and critical loads with different length parameter l

5.2 Validity of the model

It is well known that the strain gradient model is suitable to describe the size effects in a microscopic scale in some materials. The main parameter model is the characteristic length l. This length parameter represents the influence of the microscopic scale of the material heterogeneities such as the distance of atoms in crystal, the average chain length in polymers, and the average size of grains in alloys and so on. In the literature, the measurement of this parameter is often based on the materials response at a microscopic scale. These works indicate that the characteristic length scale for non-local effects is at the order of several nanometres for pure metals whereas it may be larger for polymers and composites. In this work, we realize that this parameter for PMMA materials $l \approx 0.5mm$. This value provides a good agreement between test results and model predictions. But this value is so much larger than values usually measured for polymers.

In order to explain these results, we return at the characteristic physical of length parameter. Many researches in the literature support this argument. Ravi-chandar and Yang have studied the dynamic fracture in several polymers including PMMA. They showed the existence of a large number of micro cracks before final failure in the PMMA under monotone or cyclic loading. Therefore, the heterogeneity increases in the damaged PMMA due to the nucleation and the growth of the secondary micro cracks. Thus the characteristic length l in the damaged of PMMA cans become much larger.

In short, we introduce the second gradient into the cohesive model. We can notice two complementary effects. First, the strain gradient modifies the cohesive energy constitution. The cohesive energy needed to separate the two surfaces of the crack includes not only the part due to the displacement jump, but also the part due to the displacement gradient. The second effect with the introduction is that the material becomes stiffer. As a consequence, the material near a stress concentration source is stiffer that under a uniform traction.

And we have also attempted to assess the influence of the default (holes, inclusions) size on the fracture behaviour of a brittle material. Experiments showed that this influence obviously not only occurs at a microscopic scale, but also at a macroscopic scale. This size dependence can not be well described by conventional fracture theories. The introduction of the strain gradient theory seems to be an appropriate approach.

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