THEORY AND APPLICATIONS OF WEIGHT FUNCTIONS FOR CRACKS IN PIEZOELECTRICS

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Abstract

The weight function in fracture mechanics relates the stress intensity factor at the tip of a crack in an elastic body to a point load at an arbitrary location. For a piezoelectric material, this definition is extended to include the effect of point charges and the presence of an electric displacement intensity factor at the crack tip. Applying the principle of linear superposition and BETTIs theorem of reciprocity, the weight functions for the different crack opening Modes are derived from any known mixed-mode solution in terms of displacements and electric potentials of the cracked body under specific electromechanical loads. Furthermore, another type of weight functions is derived, relating displacements and electric potentials in a cracked body to the field intensity factors. As an example, both types of weight functions are calculated for the GRIFFITH crack in a piezoelectric material.

Introduction

Fracture mechanics of piezoelectric and ferroelectric materials is of increasing interest for applicants of smart materials. The analytical framework of piezoelectric fracture mechanics has been established in the early 90s by researchers like Sosa [1], Pak [2] or Park and Sun [3] giving rise to an increasing interest of scientists in this field. Anyhow, the concept of crack weight functions, well known in fracture mechanics of classical materials since 1970 (Bueckner [4]) has only recently been extended to piezoelectric materials by Ma and Chen [5] as well as McMeeking and Ricoeur [6]. In [5] the derivation of the weight function is based on a work-conjugate integral following Bueckner, in [6] the derivation goes back on a procedure published by Rice [7] leading to a more transparent formulation.

The crack weight functions derived in this article are twofold. The first type relates the stress and electric displacement intensity factors to point loads and charges acting at an arbitrary location of the piezoelectric body. The second type is used to calculate field intensity factors due to local displacements/electric potentials located anywhere in the cracked body. The weight functions for cracks in piezoelectric materials are formulated from MAXWELL relationships among the energy release rate with displacements and the electric potential (type 1) or stresses and charge densities (type 2) as dependent variables and applied loads and electric charges (type 1) or displacemens and electric potentials (type 2) as independent variables.

Applying the principle of linear superposition and BETTIs theorem of reciprocity, the weight functions for a body with a crack are calculated from any known mixed-mode solution of the boundary value problem. Thus, knowing the solution for one specific set of boundary conditions, every mixed boundary value problem of the body under consideration can be solved in terms of *K*-factors by integrating weight functions and electromechanical loads along the DIRICHLET and NEUMANN boundaries. Results are presented for the GRIFFITH crack, calculating weight functions for point loads on the crack faces and for dislocations on the ligament.

Fundamental relations of piezoelectric fracture mechanics

The mathematical fundamentals of piezoelectric fracture mechanics can be studied in early articles (see e.g. [1,2,3]) or lately published work reviewing the field as Zhang *et al.* [8] or Qin [9]. The phenomenological description of the field problem under quasistatic loading is governed by the balance equations of linear elasticity and electrostatics:

$$\begin{aligned} \sigma_{ij,j} + b_i &= 0 \\ D_{j,j} - \omega_V &= 0 \end{aligned} \tag{1}$$

with the stress tensor σ_{ij}, the electric displacements D_j, volume forces b_i and volume charges denoted as ω_V. Being the unit normal vector n_j of an arbitrary plane within the piezoelectric body or at its boundary, mechanical stresses t_i and electric surface charge densities ω_S can be written as

$$\begin{aligned} \sigma_{ij}\, n_j &= t_i \\ D_j\, n_j &= -\omega_S \end{aligned} \tag{2}$$

The strain tensor ε_{ij} is deduced from the displacement vector u_i as

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \tag{3}$$

and the electric field vector E_j is derived from the electric potential φ as

$$E_j = -\varphi_{,j} \tag{4}$$

In piezoelectric materials mechanical and electrical fields are coupled, which is described by the constitutive equations of piezoelectricity:

$$\begin{aligned} \sigma_{ij} &= C_{ijkl}\, \varepsilon_{kl} - e_{mij}\, E_m \\ D_n &= e_{nkl}\, \varepsilon_{kl} + \kappa_{nm}\, E_m \end{aligned} \tag{5}$$

where C_{ijkl}, e_{mij} and κ_{nm} denote the tensors of elastic, piezoelectric and dielectric constants. C_{ijkl} and κ_{nm} are symmetric and positive definite. In fracture mechanics of piezoelectrics an additional electric displacement intensity factor K_{IV} is introduced, following the definition of stress intensity factors in classical fracture mechanics:

$$K_{IV} = \lim_{r\to 0} \sqrt{2\pi r}\; D_2(\theta = 0) \tag{6}$$

with $D_2(\theta=0)$ denoting the electric displacement on the ligament. The asymptotic crack tip fields ($r \to 0$) can be written as

$$\begin{aligned}\sigma_{ij}(r,\theta) &= \frac{1}{\sqrt{2\pi r}}\left[K_I\ f_{ij}^{I}(\theta)+K_{II}\ f_{ij}^{II}(\theta)+K_{III}\ f_{ij}^{III}(\theta)+K_{IV}\ f_{ij}^{IV}(\theta)\right]\\ D_j(r,\theta) &= \frac{1}{\sqrt{2\pi r}}\left[K_I\ g_{j}^{I}(\theta)+K_{II}\ g_{j}^{II}(\theta)+K_{III}\ g_{j}^{III}(\theta)+K_{IV}\ g_{j}^{IV}(\theta)\right]\end{aligned} \tag{7}$$

with the polar angle θ around the crack tip and the angular functions $f_{ij}(\theta)$ and $g_j(\theta)$. Introducing an electromechanical energy release rate *G*, a generalized IRWIN relationship can be formulated connecting *G* to the field intensity factors:

$$G = -\frac{d\,\Pi}{d\,A} = \frac{1}{2}\{K\}^T[Y]\{K\} = \frac{1}{2}\left(K_I, K_{II}, K_{III}, K_{IV}\right)\begin{pmatrix} \frac{1}{c_T} & 0 & 0 & \frac{1}{e}\\ 0 & \frac{1}{c_L} & 0 & 0\\ 0 & 0 & \frac{1}{c_A} & 0\\ \frac{1}{e} & 0 & 0 & -\frac{1}{\kappa}\end{pmatrix}\begin{pmatrix}K_I\\K_{II}\\K_{III}\\K_{IV}\end{pmatrix} \tag{8}$$

with the IRWIN matrix [*Y*] and the total potential energy Π stored in the body.

Piezoelectric crack weight functions

Definitions of the piezoelectric weight functions

The weight functions $[h]$ connecting point forces/charges $\{F\}$ at the location $\vec{x}$ in a body with a crack of length ℓ to field intensity factors $\{K\}$ are defined as

$$\{K\} = \begin{pmatrix}K_I\\K_{II}\\K_{III}\\K_{IV}\end{pmatrix} = \begin{pmatrix} h_{I1} & h_{I2} & h_{I3} & h_{I4}\\ h_{II1} & h_{II2} & h_{II3} & h_{II4}\\ h_{III1} & h_{III2} & h_{III3} & h_{III4}\\ h_{IV1} & h_{IV2} & h_{IV3} & h_{IV4}\end{pmatrix}\begin{pmatrix}F_1\\F_2\\F_3\\Q\end{pmatrix} = \left[h(\vec{x},\ell)\right]\ \left\{F\left(\vec{x}\right)\right\} \tag{9}$$

In Fig. 1 an arbitrary piezoelectric body with an edge crack subject to point loads ($\vec{F}$,Q)^T and generalized surface tractions $\{t\} = \left(\vec{t}, \omega_S\right)^T$ is shown. The field intensity factors due to line loads $\{t\}$ distributed on a NEUMANN type boundary S_t are calculated by integration:

$$\{K\} = \int_{S_t}[h]\{t\}\,dS \tag{10}$$

Eqs. (9) and (10) define weight functions of the type 1. The second type of weight functions relates generalized displacements $\{u\}$ to the field intensity factors:

$$\{K\}=\left[h^u\left(\vec{x},\ell\right)\right]\begin{pmatrix}u_1\\u_2\\u_3\\\varphi\end{pmatrix}=\left[h^u\left(\vec{x},\ell\right)\right]\left\{u\left(\vec{x}\right)\right\} \tag{11}$$

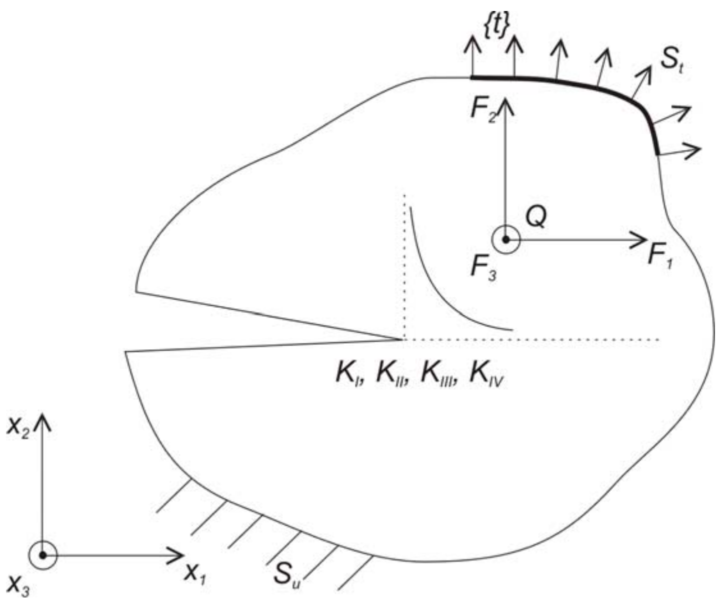

FIGURE 1. Piezoelectric body with an edge crack subject to point loads and electromechanical surface tractions.

Prescribing displacements and electric potentials along a DIRICHLET type boundary S_u, see Fig. 1, *K*-factors are calculated by integration:

$$\{K\}=\int_{S_u}\left[h^u\right]\{u\}\,dS \tag{12}$$

If both types of weight functions $[h]$ and $\left[h^u\right]$ are known for a specific body with a crack, a general mixed boundary value problem with prescribed tractions/charges on the part of the boundary S_t and displacements/electric potentials on S_u can be solved in terms of *K*-factors superimposing Eqs. (10) and (12).

Derivation of type 1 weight functions $[h]$

In the following two different types of loads are considered a body can be exposed to: (1) a point load $\{F\}$ and (2) a distributed boundary load acting on the structure. The latter will be referred to as “reference load” in the following. According to the superposition principle the field intensity factors can be expressed as

$$\{K\} = \{K\}^{(2)} + [h]\{F\} \tag{13}$$

where $\{K\}^{(2)}$ denotes the four intensity factors due to the reference load (2) and the weight functions $[h]$ are introduced by Eq. (9). Assuming the validity of BETTIs theorem of reciprocity, the principle of CASTIGLIANO can be formulated

$$\frac{\partial W}{\partial \{F\}} = \begin{pmatrix} \partial W/\partial F_1 \\ \partial W/\partial F_2 \\ \partial W/\partial F_3 \\ \partial W/\partial Q \end{pmatrix} = \{u\}^{(1)} \tag{14}$$

equating the partial derivatives of the work of external loads W with respect to generalized point forces $\{F(\vec{x})\}$ to generalized displacements at the location $\vec{x}$, i.e. $\{u\}^{(1)}$. Differentiating both sides of Eq. (14) with respect to the crack length ℓ and introducing the energy relase rate G according to Eq. (8) gives

$$\frac{\partial W}{\partial \{F\}\,\partial \ell} = \frac{\partial G}{\partial \{F\}} = \frac{\partial \{u\}^{(1)}}{\partial \ell} \tag{15}$$

The energy release rate is related to the field intensity factors of the combined problem, see Eq. (13), applying the IRWIN relationship, see Eq. (8). Taking into account the symmetry of the IRWIN matrix $[Y]$, the derivative with respect to $\{F\}$ can be written as

$$\frac{\partial G}{\partial \{F\}} = [h]^T [Y]\left(\{K\}^{(2)} + [h]\{F\}\right) \tag{16}$$

Eqs. (15) and (16) result in a linear system of equations for the calculation of the unknown weight functions, if the point loads $\{F\}$ in Eq. (16) are set to zero:

$$\left[h(\vec{x},\ell)\right]^T \{K\}^{(2)} = [Y]^{-1} \frac{\partial \{u(\vec{x},\ell)\}^{(1)}}{\partial \ell} \tag{17}$$

In general, to determine the 16 unknown weight functions at a location $\vec{x}$ in a body with the crack length ℓ, four different reference loads have to be chosen leading to four linear independent sets of intensity factors $\{K\}^{(2)}$ and to four generalized displacements $\{u\}^{(1)}$ at $\vec{x}$. Analytical solutions e.g. for the GRIFFITH crack may directly yield all weight functions in a closed-form.

Derivation of type 2 weight functions $[h^u]$

As in the previous section, we consider the two loading conditions (1) and (2), where the reference load (2) again represents an arbitrary electromechanical (mostly boundary) load leading to known intensity factors. With $[C]$ being a symmetric stiffness matrix, generalized displacements and tractions can be related to each other

$$\begin{aligned}\{T\} &= [C]^{(1)}\{u\}^{(1)} + [C]^{(12)}\{u\}^{(2)} \\ \{t\} &= [C]^{(21)}\{u\}^{(1)} + [C]^{(2)}\ \{u\}^{(2)}\end{aligned} \tag{18}$$

where $\{u\}^{(2)}$ denotes the displacements/electric potentials on the boundary where the distributed loads (2) are acting. The problem (1) is defined by point loads $\{T\}$ having the same units as the generalized stresses $\{t\}$. The superscripts at the stiffness matrices refer to the load cases (1) and (2). Due to BETTIs theorem of reciprocity the cross stiffnesses $[C]^{(12)}$ and $[C]^{(21)}$ are equal. Making use of Eq. (18), the work of the external forces W can be calculated:

$$\begin{aligned}W &= \frac{1}{2}\{t\}^T\{u\}^{(2)} + \frac{1}{2}\{T\}^T\{u\}^{(1)} = \frac{1}{2}\{u\}^{(1)T}[C]^{(1)}\{u\}^{(1)} + \\ &\quad + \{u\}^{(2)T}[C]^{(12)}\{u\}^{(1)} + \frac{1}{2}\{u\}^{(2)T}[C]^{(2)}\{u\}^{(2)}\end{aligned} \tag{19}$$

Forming the derivative of the work W with respect to the generalized displacements $\{u\}^{(1)}$ and accounting for Eq. (18) yields

$$\frac{\partial W}{\partial\{u\}^{(1)}} = [C]^{(12)}\{u\}^{(2)} + [C]^{(1)}\{u\}^{(1)} = \{T\} \tag{20}$$

Eq. (20) has a similar structure as Eq. (14) corresponding to CASTIGLIANOs principle. Therefore, a relationship between the energy release rate G and point forces, similar to that in Eq. (15) can be derived:

$$\frac{\partial W}{\partial\{u\}^{(1)}\partial\ell} = \frac{\partial G}{\partial\{u\}^{(1)}} = \frac{\partial\{T\}}{\partial\ell} \tag{21}$$

According to the definition of the weight functions $[h^u]$ in Eq. (11), the field intensity factors for a superposition of the two load cases are

$$\{K\} = \{K\}^{(2)} + \left[h^u\right]\{u\}^{(1)} \tag{22}$$

The derivative of the energy release rate with respect to the displacements at the location of the point force is determined from Eq. (8) and Eq. (22):

$$\frac{\partial G}{\partial\{u\}^{(1)}} = \left[h^u\right]^T[Y]\left(\{K\}^{(2)} + \left[h^u\right]\{u\}^{(1)}\right) \tag{23}$$

The unknown weight functions (type 2) are calculated from a linear set of equations accounting for Eq. (21) and assuming vanishing displacements and electric potentials $\{u\}^{(1)}$ on the DIRICHLET boundary where $[h^u]$ is to be determined:

$$\left[h^u(\vec{x},\ell)\right]^T\{K\}^{(2)} = [Y]^{-1}\frac{\partial\{T(\vec{x},\ell)\}}{\partial\ell} \tag{24}$$

Again, four different reference loads have to be chosen leading to four linear independent sets of intensity factors $\{K\}^{(2)}$ and to four generalized stresses $\{T\} = (\vec{t}, \omega)^T$ at $\vec{x}$.

Weight functions for the GRIFFITH crack

The considerations in this section are restricted to the crack plane as reference plane for the calculation of weight functions. Displacements and electric potentials on the faces of a GRIFFITH crack of length ℓ with an electromechanical load $\{t\}$ on the crack faces as reference load (2) are (see e.g. Ricoeur and Kuna [10])

$$\{u\}^{(1)} = \begin{pmatrix} \vec{u} \\ \varphi \end{pmatrix} = \pm[Y]\{t\}\sqrt{\frac{\ell^2}{4} - x_1^2} = \pm[Y]\{K\}^{(2)}\sqrt{\frac{\ell^2 - 4x_1^2}{2\pi\ell}} \tag{25}$$

if x_1 is the coordinate along the crack faces with its origin in the center of the crack. The stresses in front of the crack tip in the plane of the ligament are

$$\{T\} = \frac{x_1}{\sqrt{x_1^2 - (\ell/2)^2}}[I]\{t\} = \frac{1}{\sqrt{\pi(\ell/2)}}\frac{x_1}{\sqrt{x_1^2 - (\ell/2)^2}}[I]\{K\}^{(2)} \tag{26}$$

with the identity matrix $[I]$. The type 1 weight functions for both crack faces considering the right-hand crack tip are calculated from Eq. (17):

$$[h] = \pm\frac{1}{\sqrt{\pi\ell/2}}[I]\frac{\partial}{\partial\ell}\sqrt{\frac{\ell^2}{4} - x_1^2} = \pm\frac{1}{2\sqrt{\pi\ell/2}}\sqrt{\frac{\ell/2 + x_1}{\ell/2 - x_1}}[I] \tag{27}$$

The type 2 weight functions for point dislocations in terms of displacements and electric potentials on the ligament are calculated from Eqs. (24) and (26):

$$[h^u] = \frac{1}{\sqrt{\pi\ell/2}}[Y]^{-1}\frac{\partial}{\partial\ell}\frac{x_1}{\sqrt{x_1^2 - (\ell/2)^2}} = \frac{1}{\sqrt{8\pi\ell}}[Y]^{-1}\frac{\ell}{(x_1 - \ell/2)\sqrt{x_1^2 - (\ell/2)^2}} \tag{28}$$

Summary

Weight functions for cracks in piezoelectrics have been derived. The first type quantifies the influence of point forces and charges on the field intensity factors at the crack tip. The second type calculates intensity factors due to displacements and electric potentials. As an example, both types are calculated for the GRIFFITH crack.

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