LIMIT EQUILIBRIUM OF A FINITE CYLINDRICAL SHELL WITH SURFACE CRACK

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Abstract

A closed cylindrical shell is weakened by a longitudinal surface (external or internal) crack. The shell material is taken to be ideally elasto-plastic or strengthening. The external load, crack size and material properties are assumed such that plastic strains develop like a narrow strip on the crack extension through the whole crack thickness. According to the δ_c-model analogue the plastic strain strip is replaced by the surface of displacements and rotation angles discontinuity, and the plastic zone response to the elastic volume is changed for unknown forces and moments. Using the method of generalized coupling problems, the finite shell is replaced by an infinite one taking into account the boundary conditions at the ends of the initial finite shell. Thus, the $3D$ elasto-plastic problem is reduced to the $2D$ one on elastic equilibrium of the infinite shell with a through crack of unknown length, to which faces the unknown forces and moments are applied. The method for solving the elastic problem consists in obtaining a system of eight singular integral equations (four at the imaginary crack line and four at the shell ends $\alpha = \pm l_e / R$) with unknown limits of integration and discontinuous right-hand sides. The algorithm of numerical solving the obtained system under the plasticity conditions for thin shells and conditions of forces and moment boundedness near the crack is proposed. For the shell under the internal pressure, rigidly fixed at the shell ends or hinge-supported, the crack front opening and plastic zone dependences on the geometric and mechanical parameters are analyzed.

INTRODUCTION

Real structures contain different surface, internal or through defects (cracks, pores, inclusions, cuts etc.) having various geometry (nonregular as a rule). The faulty zone may contain several internal cracks of arbitrary shape. When developing the calculation schemes, these defects are replaced by cracks as the most dangerous ones from the strength viewpoint. It is assumed that these cracks are of ideal shape, have equivalent sizes and orientation. The calculation schemes for such replacement are given in methodical recommendations [1] concerning bodies of reactors and steam generators, pipelines, rotors of turbines and turbo-generators, and other energy equipment to be calculated from the standpoint of crack growth resistance. The geometric parameters of structure elements and defects as well as the parameters of the stressed state are schematized.

It should be noticed that even for an idealized non-through crack in thin-walled structure elements, the problem on stress distribution in its vicinity is very complicated because of three-dimensionality and a need for consideration of plastic strains. Therefore, for solution of such problems an approximate model based on the analogue of the δ_c-model was proposed in Refs. [2-4]. In the abovementioned references, the unlimited ideally elasto-plastic shells were examined. In this work we consider the limited shell, and its material is taken to be strengthening.

BASIC EQUATIONS AND RELATIONS

Consider a closed cylindrical shell of the thickness $2h$ and the length $2l_e$ referred to a three-orthogonal coordinate system (α, β, γ), where $\alpha = z/R$ (R is the radius of the shell median surface, z is the distance along the generatrix), $\beta = y/R$ (y is the distance along the directrix), γ is a coordinate normal to the median surface. The shell is weakened by an internal longitudinal crack at $\beta = 0$; $|\alpha| < \alpha_0$; $-h + 2d_1 \le \gamma \le h - 2d_2$ (Fig.1) ($\alpha_0 = l_0/R$, $2l_0$ is the crack length, $2(h - d_1 - d_2)$ is the crack depth). Having assumed $d_1 = 0$, we obtain an internal surface crack, and for $d_2 = 0$ an external surface one.

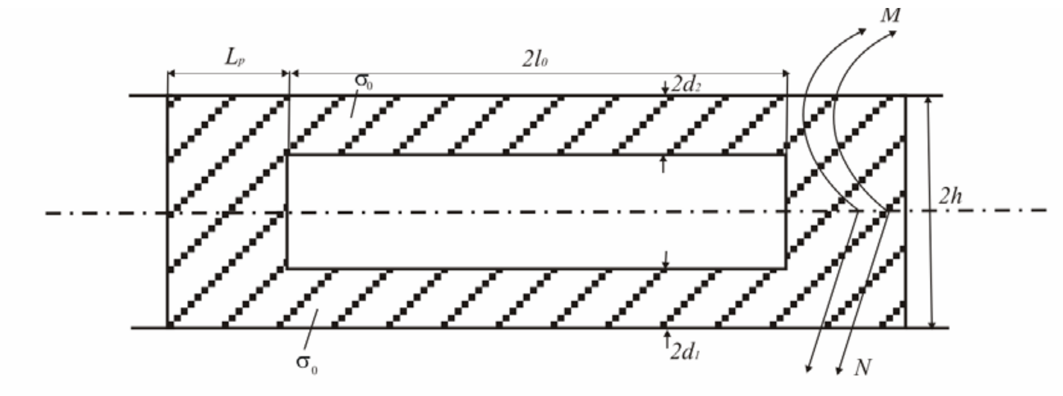

FIGURE 1.

Model of Elastico-Plastic Shell with Internal Crack

Assume that the shell stressed state is symmetric with respect to the surfaces $\alpha = 0$ and $\beta = 0$. We shall restrict ourselves to consideration of sufficiently deep cracks ($d_1 + d_2 \le 0{,}6h$). The crack sizes, the level of external loading and material properties are assumed such that plastic strains in the crack vicinity extend through the whole shell depth as a narrow strip. Now, according to the analogue of the δ_c-model [5], the zone of plastic strains is replaced by the surface of elastic displacements and rotation angles discontinuity, and the reaction of plastic zone material to elastic volume is changed for corresponding forces and moments. Assume that constant stresses $\sigma^0 = (\sigma_B + \sigma_T)/2$ (σ_B is the strength limit, and σ_T is the yield point of the shell material) act on the crack extension in depth to the external or internal shell surfaces, i. e. in the region $\alpha \in \left]-\alpha_0, \alpha_0\right[$, $\gamma \in [-h, -h + 2d_1] \cup [h - 2d_2, h]$. In plastic zones (on the crack extension along the length) the unknown normal forces N and bending moment M act. They satisfy the plasticity condition for thin shells [5].

Thus, within the scope of assumed model the elasto-plastic problem on a non-through crack of the length $2l_0$ is replaced by the elastic problem on a through crack of unknown length $2l_1 = 2(l_0 + l_p)$, where l_p is the unknown length of plastic zone. On the faces of this crack the following conditions for the components of perturbed stressed state are satisfied:

$$N_2 = \begin{cases} N^l - N_2^0, & |\alpha| < \alpha_0; \\ N - N_2^0, & \alpha_0 \le |\alpha| \le \alpha_1; \end{cases}$$

$$M_2 = \begin{cases} M^l - M_2^0, & |\alpha| < \alpha_0; \\ M - M_2^0, & \alpha_0 \le |\alpha| \le \alpha_1 \end{cases} \quad (1)$$

Here N^l, M^l are the normal force and the bending moment being the response of the material to a break of internal bonds over and under the crack, $\alpha_1 = l_1 / R$. The normal force and the bending moment, according to the assumptions adopted for the stressed state in these problems, are defined according to the formulas

$$N^l = 2(d_1 + d_2)\sigma^0; \qquad M^l = 2\sigma^0(h - d_1 - d_2)(d_2 - d_1);$$

where N_2^0, M_2^0 are the force and the moment at the crack line in a continuous shell caused by external loadings.

Modelling of a Finite Shell

The solution to the elastic problem is constructed by the method of generalized conjugation problems [5, 6] for modelling a finite shell by an infinite one with analogous mechanical and geometric parameters for $|\alpha| < \alpha_l$ and by the distortion method [5, 7] for modelling a through crack $|\alpha| < \alpha_1$; $\beta = 0$ by internal stress sources with unknown densities. As this takes place, using the representation

$$p(\alpha) = p_k(\alpha)\chi(\alpha) \quad (2)$$

a system of differential equilibrium equations in displacements, extended on the whole region occupied by the unlimited shell, has the following form:

$$L_{ij}u_j = g_i'(\alpha,\beta) + g_i''(\alpha,\beta), \quad i, j = \overline{1,3}. \quad (3)$$

In relations (2) and (3) $p(\alpha)$ and $p_k(\alpha)$ are the unknown and given functions for infinite and finite shell, respectively; $\chi(\alpha) = S_-(\alpha + \alpha_e) - S_-(\alpha - \alpha_e)$ is the characteristic function for the region occupied by the finite shell; $S_-(\alpha)$ is the asymmetric unit function [6]; L_{ij} are the differential operators analogous to those for a homogeneous shell, but in a case under consideration according to representation of Poisson's ratio $\nu(\alpha)$ in the form (2), they have discontinuous coefficients; u_j are the displacement components of the shell median surface; $g_i'(\alpha,\beta)$ are the internal stress sources expressed in terms of the generalized functions and jumps of displacement and rotation angle across the crack line

$$g_1'(\alpha,\beta) = \nu(\alpha)\frac{\partial}{\partial\alpha}\varepsilon_{22}^0; \qquad g_2'(\alpha,\beta) = \frac{\partial}{\partial\beta}\varepsilon_{22}^0 + Rc_1^2\frac{\partial}{\partial\beta}\kappa_{22}^0;$$

$$g_3'(\alpha,\beta) = \varepsilon_{22}^0 - Rc_1^2\left(\nu(\alpha)\frac{\partial^2}{\partial\alpha^2} + \frac{\partial^2}{\partial\beta^2}\right)\kappa_{22}^0; \quad (4)$$

$$\varepsilon_{22}^{0} = [u_2(\alpha)]_c \chi(\alpha)\delta(\beta); \qquad \kappa_{22}^{0} = [\theta_2(\alpha)]_c \chi(\alpha)\delta(\beta)$$

$g_i''(\alpha,\beta)$ are expressed in terms of generalized functions and jumps of derivatives of the displacement components across the surfaces $\alpha = \pm\alpha_e$:

$$g_1'' = \left[\frac{\partial}{\partial\alpha}u_1\right]_k \delta_-(\alpha-\alpha^*); \qquad g_2'' = \frac{1}{2}(1+4c_1^2)(1-\nu(\alpha))\left[\frac{\partial}{\partial\alpha}u_2\right]_k \delta_-(\alpha-\alpha^*);$$

$$g_3'' = c_1^2\left\{\left[\frac{\partial^3}{\partial\alpha^3}u_3\right]_k - (2-\nu(\alpha))\frac{\partial}{\partial\beta}\left[\frac{\partial}{\partial\alpha}u_2\right]_k\right\}\delta_-(\alpha-\alpha^*) + \left[\frac{\partial}{\partial\alpha}u_3\right]_k \delta'(\alpha-\alpha^*); \quad (5)$$

$\delta(\beta)$ is the Dirac delta function; $\qquad \chi(\alpha) = S(\alpha+\alpha_1) - S_-(\alpha-\alpha_1)$;

$[p(\alpha)]_e = p(\alpha,+0) - p(\alpha,-0)$; $[p(\alpha)]_k = p(\alpha^*+0,\beta) - p(\alpha^*-0,\beta)$;

$$\theta_2(\alpha) = R^{-1}\left(\frac{\partial}{\partial\beta}u_3 - u_2\right); \qquad \alpha^* = \begin{cases} -\alpha_e & \text{for the left end} \\ +\alpha_e & \text{for the right end} \end{cases}.$$

Procedure of Obtaining Integral Equations

On the basis of the fundamental solution of the system (3) and the convolution operation, the integral representations for the parameters of the stressed-strained state are constructed. These representations are expressed in terms of six unknown jumps of displacement and their derivatives.

$$X^{(i)}(\alpha,\beta) = \sum_{j=1}^{4}\int_{-\pi}^{\pi}\Omega_j(\theta)Q^{ij}(\alpha,\theta-\beta)d\theta + \sum_{j=5}^{6}\int_{-\alpha_1}^{\alpha_1}\Omega_j(\xi)Q^{ij}(\xi-\alpha,\beta)d\xi, \quad i=\overline{1,6}; \qquad (6)$$

Here

$$\Omega_1(\beta) = K_E\left[\frac{\partial}{\partial\alpha}u_1\right]_k, \quad \Omega_2(\beta) = K_E Rc\left[\frac{\partial}{\partial\alpha}u_2\right]_k, \quad \Omega_3(\beta) = K_E\left[\frac{\partial}{\partial\alpha}u_3\right]_k,$$

$$\Omega_4(\beta) = K_E Rc\left[\frac{\partial^3}{\partial\alpha^3}u_3\right]_k, \quad \Omega_5(\beta) = K_E[u_2(\alpha)]_c, \quad \Omega_6(\beta) = K_E Rc[\theta_2(\alpha)]_c,$$

$$X^{(i)}(\alpha,\beta) = K_E R^{-1}u_i(\alpha,\beta), \quad i=\overline{1,3}, \quad X^{(4)}(\alpha,\beta) = K_E\theta_1(\alpha,\beta),$$

$$X^{(5)}(\alpha,\beta) = -N_2(\alpha,\beta)/(2h\sigma^0), \quad X^{(6)}(\alpha,\beta) = -k^*M_2(\alpha,\beta)/(h^2\sigma^0),$$

$$K_E = -E/(\pi\sigma_0), \quad k^* = h/(2Rc), \quad c^2 = h^2/(3(1-\nu^2)R^2)$$

$Q^{ij}(\alpha,\beta)$ are combinations of derivatives of the fundamental solution to the system of differential equations (3).

Integral representation (6) expresses the components of disturbed (due to a crack) stressed-strained state of infinite shell in terms of six unknown functions – the densities of internal stress sources, two of which $\Omega_i(\alpha)$ $(i=5,6)$ are concentrated along the segment $|\alpha| < \alpha_1$, $\beta = 0$ and the rest four $\Omega_i(\beta)$ $(i=\overline{1,4})$ are concentrated along the segment $\alpha = -\alpha_e$, $|\beta| \le \pi$.

Let $X^{i,1}$ and $X^{i,0}$ be the components of total and principal stressed-strained state of the shell, respectively, and $X^{(i)}$ are the components of the perturbed one. Then, according to the superposition principle,

$$X^{i,1} = X^{(i)} + X^{i,0}, \tag{7}$$

where $X^{i,0}$ is a particular solution to the problem on the stressed state of infinite shell without a crack under external load acting in the region $|\alpha| < \alpha_e$, $|\beta| \le \pi$. This solution is assumed to be known. If the unknown densities $\Omega_i \quad (i = \overline{1,6})$ satisfying the boundary conditions at the ends $\alpha = \pm\alpha_e$

$$X^{(i)}(-\alpha_e + 0, \beta) + X^{i,0}(-\alpha_e, \beta) = X^{ie}(\beta), \quad i = \overline{1,4} \tag{8}$$

and conditions (1) at the crack faces are found, the problem is solved. In conditions (8) X^{ie} are the corresponding parameters of the stressed-strained state given at the end $\alpha = -\alpha_e$. If conditions (8) are satisfied, the analogous conditions at the end $\alpha = \alpha_e$ are satisfied as a consequence.

If we substitute integral representation (6) into the boundary conditions (8) and (1), the system of six integral equations for determining the functions $\Omega_i \quad (i = \overline{1,6})$ is obtained.

$$\sum_{j=1}^{4} \int_{-\pi}^{\pi} \Omega_j(\theta) q^{ij}(\theta - \beta) d\theta + \sum_{j=5}^{6} \int_{-1}^{1} \Omega_j(t) q^{i,j+4}(t, \beta) dt = f^i(\beta), \quad i = \overline{1,4};$$

$$\sum_{j=1}^{4} \int_{-\pi}^{\pi} \Omega_j(\theta) Q^{ij}(\theta - \beta) d\theta + \sum_{j=5}^{6} \int_{-1}^{1} \Omega_j(t) \left[\frac{a_{i,j}}{t - s} + \alpha_1 K_{ij}(\alpha_1(t - s)) \right] dt = f_{i-4}(s), \ |s| < 1, \ i = 5,6; \tag{9}$$

Here

$$q^{ij}(\beta) = \lim_{\alpha \to \alpha_e + 0} Q^{ij}(\alpha, \beta), \quad i, j = \overline{1,4};$$

$$q^{ij}(t, \beta) = \alpha_1 Q^{ij}(\alpha_1 t + \alpha_e, \beta), \quad i = \overline{1,4}, \ j = 5,6, \ |t| < 1;$$

$$f^i(\beta) = X^{ie}(\beta) + X^{i,0}(-\alpha_e, \beta), \quad f_1(s) = N_2(\alpha / \alpha_1), \quad f_2(s) = M_2(\alpha / \alpha_1);$$

$K_{ij}(z)$ are continuous functions.

Numerical Algorithm for Solving the System of Singular Equations

It should be noted that due to conditions (1), the right-hand sides of last two equations in the system (9) are discontinuous functions. Numerical experiment has shown that direct numerical methods for solving such equations give a significant error at discontinuity points. Therefore, by analogy with [8,9], the solution for function $\Omega_i \quad (i = 5,6)$ is sought in the form

$$\Omega_i(t) = h_i(t) + \Psi_i(t), \tag{10}$$

where $h_i(t)$ is the solution of corresponding canonical singular integral equation with discontinuous right-hand side, which is found using the inversion formula for the Cauchy-type integral. To find the functions $\Psi_i(t)$ we obtain a system of integral equations

$$\sum_{j=1}^{4}\int_{-\pi}^{\pi}\Omega_j(\theta)q^{ij}(\theta-\beta)d\theta+\sum_{j=1}^{2}\int_{-1}^{1}\Psi_{j+4}(t)q^{i,j+4}(t,\beta)dt = g^i(\beta), \quad |\beta|\le\pi,\ i=\overline{1,4};$$

$$\sum_{j=1}^{4}\int_{-\pi}^{\pi}\Omega_j(\theta)Q^{ij}(\alpha_1 s,\theta)d\theta+\sum_{j=1}^{2}\int_{-1}^{1}\Psi_{j+4}(t)\left[\frac{a_{ij}}{t-s}+\alpha_1 K_{ij}(\alpha_1(t-s))\right]dt=g_{i-4}(s),\ |s|<1,\ i=5,6, \quad (11)$$

Here

$$g^i(\beta)=f^i(\beta)-\sum_{j=1}^{2}\int_{-1}^{1}h_j(t)q^{i,j+4}(t,\beta)dt, \quad i=\overline{1,4},\ |\beta|\le\pi;$$

$$g_i(s)=-\alpha_1\int_{-1}^{1}\sum_{j=1}^{2}h_j(t)K_{ij}(\alpha_1(t-s))dt, \quad i=1,2.$$

It should be pointed out that the limits of integration in system (11) are unknown (the length of imaginary crack is unknown), and the right-hand sides contain the unknown force N and moment M. To find the length of plastic zone the plasticity conditions for thin shells in terms of N and M are satisfied. For ideally elasto-plastic materials the Tresca condition in the form of plastic hinge will be used:

$$\left(\frac{N}{2h\sigma_T}\right)^2+\frac{|M|}{h^2\sigma_T}=1. \quad (12)$$

For materials with strengthening we utilize the same condition in the form

$$N(\alpha)=P\left[(1-m^*)(|\alpha|-\alpha_0)/(\alpha_1-\alpha_0)+m^*\right]$$

$$M(\alpha)=H\left[(1-m^*)(|\alpha|-\alpha_0)/(\alpha_1-\alpha_0)+m^*\right] \quad (13)$$

$$\alpha_0\le|\alpha|\le\alpha_1$$

In relations (13) $m^*=\sigma_B/\sigma_T$; P, H are unknown constants, which have to satisfy the plasticity condition

$$[P/(2h\sigma^*)]^2+|H|/(h^{2\sigma^*})=1,$$

where

$$\sigma^*(\alpha)=(\sigma_T-\sigma_B)(|\alpha|-\alpha_0)/(\alpha_1-\alpha_0)+\sigma_B.$$

In order to find the unknown N and M we utilize the finiteness conditions for forces and moments near the tips of imaginary crack. One need satisfy only the zero condition for the corresponding force and moment intensity factors, i.e.

$$K_N=K_M=0 \quad (14)$$

The system (11) together with conditions (14) and (12) or (13) make a complete system of equations for finding the unknown jumps of displacements and rotation angels and their derivatives on the crack line and shell ends $\alpha=\pm\alpha_e$ as well as unknown values of N, M, l_p. Using the method of mechanical quadratures [10] and method of boundary elements, the system of integral equations (11) is reduced to a system of algebraic equations. The algorithm for its solution is as follows: in a certain way we set the value of l_p and solve a linear system of algebraic equations; using the conditions (14) we find N and M and verify the condition (12) for the case of elasto-plastic material or the condition (13) for material with strengthening. If the plasticity condition is satisfied up to a preassigned accuracy, the problem is solved; if no, then l_p is changed and the procedure is repeated.

NUMERICAL RESULTS

The crack opening at its arbitrary point is defined according to the formula $\delta(\alpha,\gamma)=[u_2(\alpha/\alpha_1)]+\gamma[\theta_2(\alpha/\alpha_1)], \quad |\alpha|<\alpha_1$,

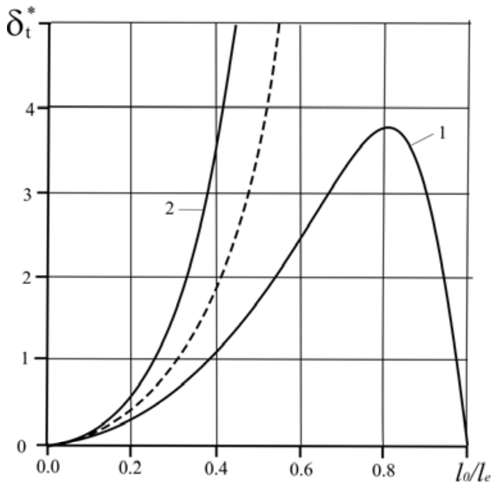

FIGURE 2.

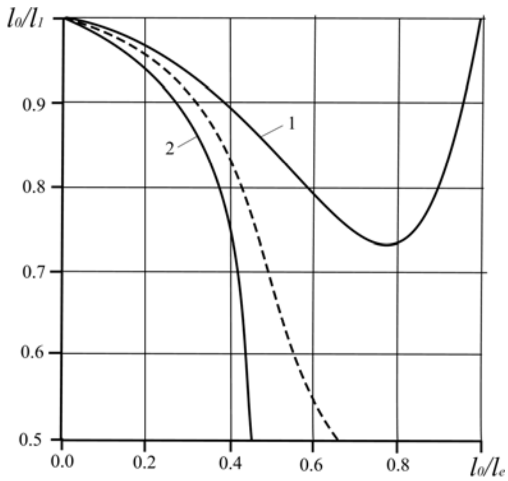

FIGURE 3.

Fig. 2 shows the surface crack opening $\delta^*=\delta E/(l_0\sigma_T)$ of depth $2d$ versus its relative length l_0/l_e in a shell under internal pressure of intensity $p\ (N^0=Rp, M^0=0)$ (the shell material is ideally elasto-plastic). Numerical analysis has been carried out for the following values of parameters: $h/R=0{,}01$; $\nu=0{,}33$; $l_e/R=0{,}8$; $n^0=Rp/(2h\sigma_T)=0{,}2$; $d/h=0{,}7$. The curve 1 corresponds to a rigid fixing of the shell ends, and the curve 2 corresponds to a hinge fixing. For comparison, the dashed line shows the results for an infinite shell for the same values of parameters h/R, ν, n^0, d/h with a crack length $l_0/R=0{,}8l_0/l_e$. The opening has been determined at the point $\alpha=\alpha_0$, $\gamma=h$. When the depth and length of the crack increase, its opening increases. For the shells made of strengthening material, the crack opening decreases when m^* increases. Fig. 3 shows the plastic zone length versus the crack length. Parameters and notations are the same as in Fig 2.

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