FAR FIELD STRESS ASYMPTOTIC BEHAVIOUR IN GROWING CREEP CRACK PROBLEMS FOR A DAMAGED MATERIAL

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Abstract

Asymptotic fields of stresses, creep strain rates and damage of mode I mode III creep cracks in steady-state growth are analyzed on the basis of Continuum Damage Mechanics. The Kachanov – Rabotnov theory is utilized and the scalar continuity parameter is incorporated into the power stress – strain rate constitutive relations. It is supposed that there is the totally damaged zone near the crack tip. The asymptotic solution for the stress and damage fields is sought outside of the totally damaged zone for large distances from the crack tip. The new far stress field determining the geometry of the totally damaged zone is found and analyzed. It can be concluded that the new far field asymptotic stress found differs from the well-known Hutchinson - Rice - Rosengren (HRR)-solution, the HRR-solution can't be used as the remote boundary condition and the HRR-field does not govern the geometry of the totally damaged zone.

Statement of the problem

The objective of this paper is to evaluate the mechanical behavior around a growing crack tip in a damaged material under creep conditions. Asymptotic fields of stresses, strain rates and damage of a mode I and mode III creep crack in steady-state growth are analyzed on the basis of Continuum Damage Mechanics. The conventional Kachanov–Rabotnov creep–damage theory is utilized and the scalar continuity (integrity) parameter is incorporated into the power stress–creep strain rate constitutive relations:

$$\dot{\varepsilon}_{ij} = (3/2)B\left(\sigma_e/\psi\right)^{n-1} s_{ij}/\psi \,, \qquad (1)$$

where $\dot{\varepsilon}_{ij}$ labels the creep strain rate, σ_e is the equivalent stress, $\sigma_e^2 = 3\left(\sigma_{rr}-\sigma_{\theta\theta}\right)^2/4 + 3\sigma_{r\theta}^2$ for plane strain conditions, $\sigma_e^2 = \sigma_{rr}^2 + \sigma_{\theta\theta}^2 - \sigma_{rr}\sigma_{\theta\theta} + 3\sigma_{r\theta}^2$ for plane stress conditions (Mode I crack), $\sigma_e^2 = \sigma_{rz}^2 + \sigma_{\theta z}^2$ for Mode III crack, σ_{ij}/ψ and s_{ij}/ψ are the effective stresses and the effective deviatoric stresses respectively (the effective stress is the stress referred to the surface that really transmits the internal forces), ψ is referred to as continuity, a scalar related to damage representing the ration between the residual effective load-carrying area of damaged material and that of the initial perfect one. The symbols n and B are the creep exponent and material constant, respectively.

There have been plenty of literature (Murakami *et al.* [1,2], Zhao and Zhang [3,4]) devoted to the analysis of the near crack tip fields coupled with elastic, elastic–plastic, fatigue and creep damage. Some of the essential aspects of the considered set of problems can be distinguished.

1. The damage field gives significant influence on the stress field near the crack tip. 2. The mathematical structure of governing equations is affected by the modeling of damage. 3. While

Hutchinson–Rice–Rosengren (HRR)–field of non-linear mechanics always shows the stress singularity at the crack tip for any finite value of the creep exponent, the preceding material damage in front of the crack decreases the stress singularity. 4. The totally damaged zone or (and) the damage process zone need to be modeled in the crack tip region.

Therefore, it is supposed that there is the totally damaged zone near the crack tip where the damage parameter reaches its critical value and all the stress tensor components are equaled to zero. The shape of the totally damaged zone is not known and should be obtained as a part of the solution. Since the totally damaged zone is modeled in the vicinity of the crack tip, the governing system of equations of the conventional Continuum Mechanics can not be formulated directly in the crack tip region. Thus, the asymptotic solution for the stress and damage fields is sought outside of the totally damaged zone for large distances from the crack tip.

Let us take a mode I creep crack extending at a constant rate in a stationary Cartesian coordinate system OX_1X_2 as shown in Fig.1 and assume that the material in the vicinity of the crack tip is in the state of plane strain or of plane stress. We can move cartesian coordinates ox_1x_2 and polar coordinates $or\theta$ with origin at the crack tip of the moving crack, where the direction x_1 and that of $\theta = 0$ are in the direction of crack extension.

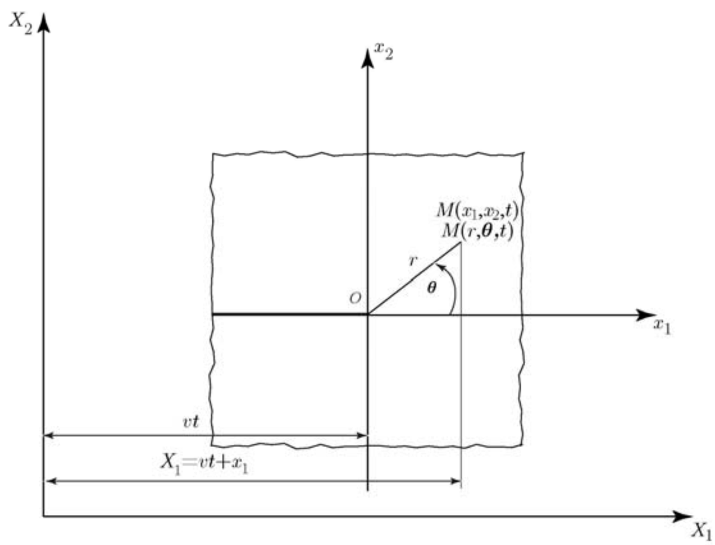

FIGURE 1. Stationary and moving coordinate systems

The governing equations for a mode I creep crack in steady-state growth are given as follows:

equilibrium equations

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0, \quad \frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + 2\frac{\sigma_{r\theta}}{r} = 0, \tag{2}$$

compatibility condition

$$2\frac{\partial}{\partial r}\left(r\frac{\partial \dot{\varepsilon}_{r\theta}}{\partial \theta} \right) = \frac{\partial^2 \dot{\varepsilon}_{rr}}{\partial \theta^2} - r\frac{\partial \dot{\varepsilon}_{rr}}{\partial r} + r\frac{\partial^2 \left(r\dot{\varepsilon}_{\theta\theta} \right)}{\partial r^2}. \tag{3}$$

We represent the damage state of material by an isotropic continuity (integrity) variable $\psi\left(0\leq\psi\leq 1\right)$ and by assuming that the creep damage is governed by the equivalent stress, the damage evolution equation in multi-axial state of stress may be given as follows (Kachanov [5], Lemaitre and Chaboche [6], Lemaitre [7])

$$d\psi/dt=-A\left(\sigma_{eqv}/\psi\right)^m, \tag{4}$$

where $A, m\ (m\approx 0.7n)$ denote material constants, $\sigma_{eqv}=\alpha\sigma_1+\beta\sigma_e+\left(1-\alpha-\beta\right)\sigma_{kk}$,

$$\frac{d}{dt}=\frac{\partial}{\partial t}-v\left(\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right) \tag{5}$$

denotes the material derivative with respect to time t. In the particular case of steady-state crack growth, we have

$$\frac{d}{dt}=-v\left(\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right) \tag{6}$$

and hence Equation (4) leads to

$$\cos\theta\frac{\partial\psi}{\partial r}-\frac{\sin\theta}{r}\frac{\partial\psi}{\partial\theta}=\frac{A}{v}\left(\frac{\sigma_{eqv}}{\psi}\right)^m \tag{7}$$

The constitutive equations (1) can be presented in the form

$$\dot\varepsilon_{rr}=-\dot\varepsilon_{\theta\theta}=\frac{3}{4}B\left(\frac{\sigma_e}{\psi}\right)^{n-1}\frac{\sigma_{rr}-\sigma_{\theta\theta}}{\psi},\quad \dot\varepsilon_{r\theta}=\frac{3}{2}B\left(\frac{\sigma_e}{\psi}\right)^{n-1}\frac{\sigma_{r\theta}}{\psi} \tag{8}$$

for plane strain conditions,

$$\begin{gathered}\dot\varepsilon_{rr}=\frac{1}{2}B\left(\frac{\sigma_e}{\psi}\right)^{n-1}\frac{2\sigma_{rr}-\sigma_{\theta\theta}}{\psi},\quad \dot\varepsilon_{\theta\theta}=\frac{1}{2}B\left(\frac{\sigma_e}{\psi}\right)^{n-1}\frac{2\sigma_{\theta\theta}-\sigma_{rr}}{\psi},\\ \dot\varepsilon_{r\theta}=\frac{3}{2}B\left(\frac{\sigma_e}{\psi}\right)^{n-1}\frac{\sigma_{r\theta}}{\psi}\end{gathered}, \tag{9}$$

for plane stress conditions. Traction – free boundary conditions are prescribed on the crack faces which require

$$\sigma_{\theta\theta}(r,\theta=\pm\pi)=0,\quad \sigma_{r\theta}(r,\theta=\pm\pi)=0\,. \tag{10}$$

The remote boundary condition can be naturally formulated in the form

$$\sigma_{ij}(r\to\infty,\theta)\to\left(\frac{C^*}{BI_nr}\right)^{1/(n+1)}\bar\sigma_{ij}(\theta,n)\,. \tag{11}$$

However, asymptotic analysis of the kinetic law of damage evolution (7) shows that it is necessary to study the eigenspectrum of a creeping damaged body with a growing crack at large distances from the crack tip. The stress and the continuity parameter are assumed in the separated form as follows

$$\sigma_{ij}(r,\theta)=r^s f_{ij}(\theta),\quad \psi(r,\theta)=1-r^{\gamma}g(\theta),\quad r\to\infty,\quad s,\gamma<0. \tag{12}$$

The cumulative damage evolution law (7) can be represented in the dimensionless form

$$\cos\theta\partial\psi/\partial r-(\sin\theta/r)\partial\psi/\partial\theta=\left(\sigma_{eqv}/\psi\right)^m . \tag{13}$$

It is obvious from Equations (12) and (13) that the stress exponent s and γ are connected by the formula $\gamma=1+sm$ whence it follows that the well-known HRR stress field (where $s=-1/(n+1)$ and $\gamma=1-m/(n+1)>0$ what contradicts the asymptotic expansions (12) and, consequently, the physical sense of the continuity parameter) can not be used as a remote boundary condition for the considered problem and the remote boundary condition is assumed to be in the form

$$\sigma_{ij}(r\to\infty,\theta)\to\tilde{C}r^s\bar{\sigma}_{ij}(\theta,n) . \tag{14}$$

Thus, it is necessary to obtain the eigenvalues s determining the far field stress asymptotic behaviour.

Asymptotic stress field

To obtain an asymptotic solution a procedure is utilized to make the equations have a dimensionless form. Dimensional analysis shows that one can introduce

$$\sigma_{ij}(r,\theta,t)=\left[\tilde{C}\left(\frac{v}{A}\right)^{\beta}\right]^{1/(\beta m+1)}\tilde{\sigma}_{ij}(\tilde{r},\theta,\tilde{t}),\quad \psi(r,\theta,t)=\tilde{\psi}(\tilde{r},\theta,\tilde{t}),$$

where $\tilde{r}=r/r_o$ and $\tilde{t}=tv/r_o$, $r_o=\left[\tilde{C}^{-m}v/A\right]^{1/(\beta m+1)}$ is considered a characteristic length (thereafter the symbol $\sim$ is omitted).

The equilibrium equations are satisfied by expressed the stress components as derivatives of the Airy stress function $F(r,\theta)$ as

$$\sigma_{\theta\theta}=\frac{\partial^2 F}{\partial r^2},\quad \sigma_{rr}=\Delta F-\sigma_{\theta\theta},\quad \sigma_{r\theta}=-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial F}{\partial\theta}\right), \tag{15}$$

$$\Delta=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} .$$

Since the present paper aims to elucidate the effect of material damage on the asymptotic fields near the creep-crack tip, we will assume the following asymptotic solution at large distances from the crack tip (at large distances from the crack tip as compared with the characteristic length of the totally damaged zone modelled in the crack tip region but at still small distances compared to the crack length, the characteristic length of the body)

$$F(r,\theta) = r^{\lambda} f(\theta) + o\left(r^{\lambda}\right), \quad \psi(r,\theta) = 1 - r^{\gamma} g(\theta) + o\left(r^{\gamma}\right),$$
$$\left(\lambda < 0, \gamma < 0, \lambda = s + 2\right) \quad r \to \infty, \tag{16}$$

where s and γ are undermined constants, $f(\theta)$ and $g(\theta)$ are unknown functions of θ. By substituting Equations (16) into Equation (15) and considering the major term of the asymptotic expansion of the stress tensor components we have the components of the asymptotic stress field as follows:

$$\begin{aligned}
\sigma_{rr}\left(r,\theta\right) &= r^{\lambda-2}\left(\lambda f + f''\right) = r^{\lambda-2} f_{rr}(\theta), \\
\sigma_{\theta\theta}\left(r,\theta\right) &= r^{\lambda-2}\lambda\left(\lambda-1\right) f = r^{\lambda-2} f_{\theta\theta}(\theta), \\
\sigma_{r\theta}\left(r,\theta\right) &= r^{\lambda-2}\left(1-\lambda\right) f' = r^{\lambda-2} f_{r\theta}(\theta)
\end{aligned} \tag{17}$$

and $\sigma_e(r,\theta) = r^{\lambda-2}\bar{\sigma}_e(\theta)$, where for plane strain conditions

$$\bar{\sigma}_e^2 = \lambda^2\left(\lambda-2\right)^2 f^2 + 4(\lambda-1)^2\left(f'\right)^2 + 2\lambda(2-\lambda) ff'' + \left(f''\right)^2, \tag{18}$$

for plane stress conditions

$$\bar{\sigma}_e^2 = \lambda^2\left(\lambda^2 - 3\lambda + 3\right) f^2 + 3(\lambda-1)^2\left(f'\right)^2 + \lambda(3-\lambda) ff'' + \left(f''\right)^2. \tag{19}$$

By substituting the stress expansions (17) into the constitutive equations (8) and, then into the compatibility equation (3), one can find the fourth order ordinary differential equation

$$\begin{aligned}
f^{IV} N(\theta) &= 4(sn+1)(1-\lambda)L(\theta) - \bar{\sigma}_e^2\left[(s\lambda + sn(sn+2)) f'' + sn(sn+2)s\lambda f\right] - \\
&-(n-1)(n-3)\left(K\left(\theta\right)/\tilde{\sigma}_e\right)^2\left(f'' - s\lambda f\right) - 2(n-1)K(\theta)\left(f''' - s\lambda f'\right) - \\
&-(n-1)M(\theta)\left(f'' - s\lambda f\right),
\end{aligned} \tag{20}$$

$$K(\theta) = \left(f'' - s\lambda f\right)\left(f''' - s\lambda f'\right) + 4\left(1-\lambda\right)^2 f'f'',$$

$$L(\theta) = (n-1)K(\theta) f' + \bar{\sigma}_e^2 f'',$$

$$M(\theta) = (f'' - s\lambda f)^2 - \left(f'' - s\lambda f\right)s\lambda f'' + 4(1-\lambda)^2 f'f'',$$

$$N(\theta) = n\left(f'' - s\lambda f\right)^2 + 4(1-\lambda)^2\left(f''\right)^2.$$

Because of the symmetry of the problem and of the vanishing condition at the crack plane we have the following boundary conditions for the non-linear forth order differential equation (20)

$$f'(\theta = 0) = 0, \quad f'''(\theta = 0) = 0, \tag{21}$$

$$f(\theta = \pi) = 0, \quad f'(\theta = \pi) = 0. \tag{22}$$

The two-point boundary value problem of Equations (20) – (22) for the asymptotic stress field $f(\theta)$ can be solved by a shooting method as an initial value problem. For this purpose, the initial values of $f(0)$ and $f''(0)$ should be specified besides the initial conditions of Equations (21). Since the differential equation (20) is homogeneous, the value of $f(0)$ can be specified arbitrary. As regards $f''(0)$, on the other hand, we can specify a proper undetermined constant c. Then, besides the condition (21), additional initial conditions

$$f(\theta = 0) = 1, \quad f''(\theta = 0) = c \tag{23}$$

are prescribed for the differential equation (20).

Thus, the boundary value problem (20), (21), (23) defines a non-linear eigenvalue problem in which s is the eigenvalue. To solve the boundary value problem numerically the Runge–Kutta–Feldberg method has been used. A shooting method has been utilised as well in order to meet the boundary condition on the crack face. The basic numerical results are presented in Table 1.

TABLE 1. Eigenvalues s obtained for mode III and mode I cracks

Mode III crack		Mode I crack (plane strain condition)		
$n = m = 1$	$s = -1.5$	$n = m = 1$	$s = -1.5$	$f''(0) = -0.75$
$n = 2,\ m = 0{,}7n$	$s = -1.2303$	$n = 2,\ m = 0{,}7n$	$s = -1$	$f''(0) = -0.5$
$n = 3,\ m = 0{,}7n$	$s = -1.1830$	$n = 3,\ m = 0{,}7n$	$s = -0.7716$	$f''(0) = -0.4372$
$n = 4,\ m = 0{,}7n$	$s = -1.1648$	$n = 4,\ m = 0{,}7n$	$s = -0.6684$	$f''(0) = -0.4092$
$n = 5,\ m = 0{,}7n$	$s = -1.1553$	$n = 5,\ m = 0{,}7n$	$s = -0.6179$	$f''(0) = -0.3985$
$n = 6,\ m = 0{,}7n$	$s = -1.1495$	$n = 6,\ m = 0{,}7n$	$s = -0.5901$	$f''(0) = -0.3958$
$n = 7,\ m = 0{,}7n$	$s = -1.1455$	$n = 7,\ m = 0{,}7n$	$s = -0.5732$	$f''(0) = -0.3950$
$n = 8,\ m = 0{,}7n$	$s = -1.1425$	$n = 8,\ m = 0{,}7n$	$s = -0.5621$	$f''(0) = -0.3948$
$n = 9,\ m = 0{,}7n$	$s = -1.1405$	$n = 9,\ m = 0{,}7n$	$s = -0.5543$	$f''(0) = -0.3943$
$n = 10,\ m = 0{,}7n$	$s = -1.1390$	$n = 10,\ m = 0{,}7n$	$s = -0.5429$	$f''(0) = -0.3940$

Note that this approach allows to consider the initially coupled boundary value problem (the continuity parameter is incorporated into the constitutive equations) as the uncoupled problem with respect to functions $f(\theta)$ and $g(\theta)$. As the major term of the continuity asymptotic expansion as $(r \to \infty)$ is supposed to be equalled to 1 that we can first analyse Equation (20) with the boundary conditions (21), (23) as it was described early. Then, having obtained $f(\theta)$ we study the ordinary differential equation with respect to $g(\theta)$ following from the damage evolution law (13)

$$\sin\theta g'(\theta) - \gamma \cos\theta g(\theta) = \sigma_{eqv}^{m}(\theta), \tag{24}$$

the right hand side of Equation (24) is the known function determined by $f(\theta)$. Equation (24) is then analyzed numerically with the regularity requirement $g(0) = -\sigma_{eqv}^{m}(0)/\gamma$. Thus, the two – term asymptotic expansion for the continuity parameter is obtained and we can find the configuration of the totally damaged zone adjacent to the crack surfaces by means of the formula $\psi = 1 - r^{\gamma} g(\theta) = 0$ and, consequently, $r(\theta) = \left(g(\theta)\right)^{-1/\gamma}$.

The obtained configuration of the totally damaged zone for observers located at different distances from the crack are shown in Figs 2-4.

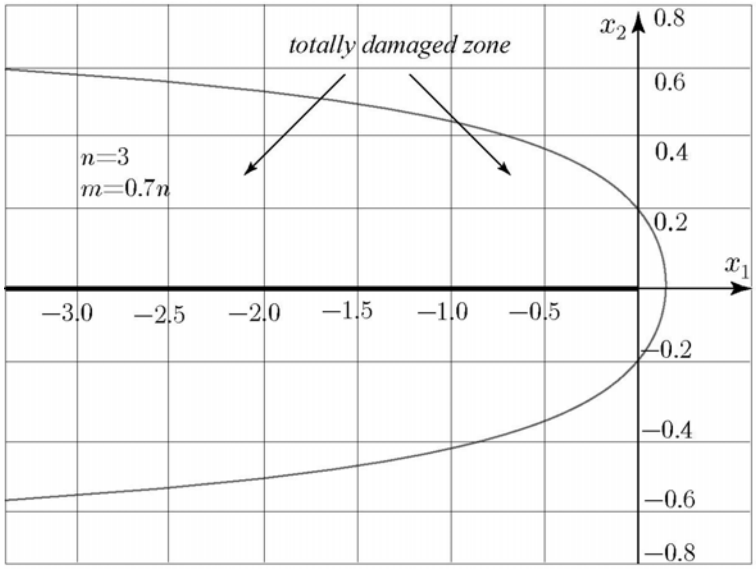

FIGURE 2. Geometry of the totally damaged zone adjacent to the crack surfaces. Mode I crack (plane strain conditions).

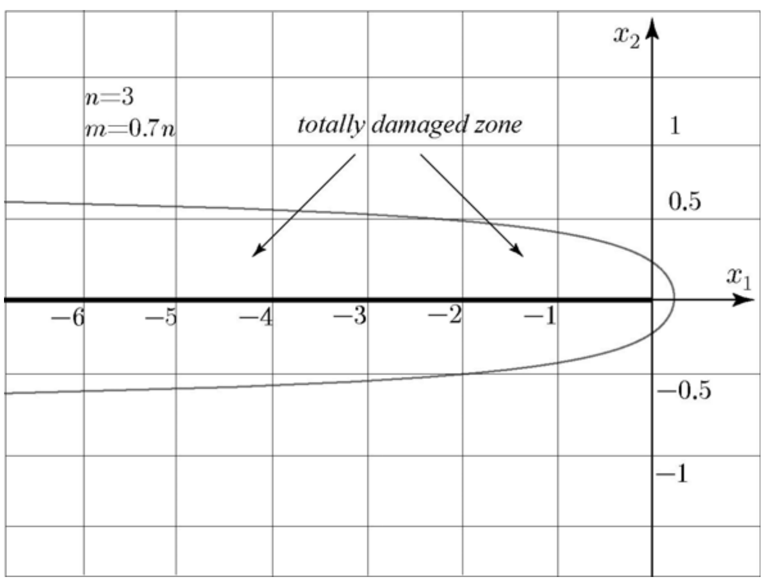

FIGURE 3. Geometry of the totally damaged zone adjacent to the crack surfaces. Mode I crack (plane strain conditions).

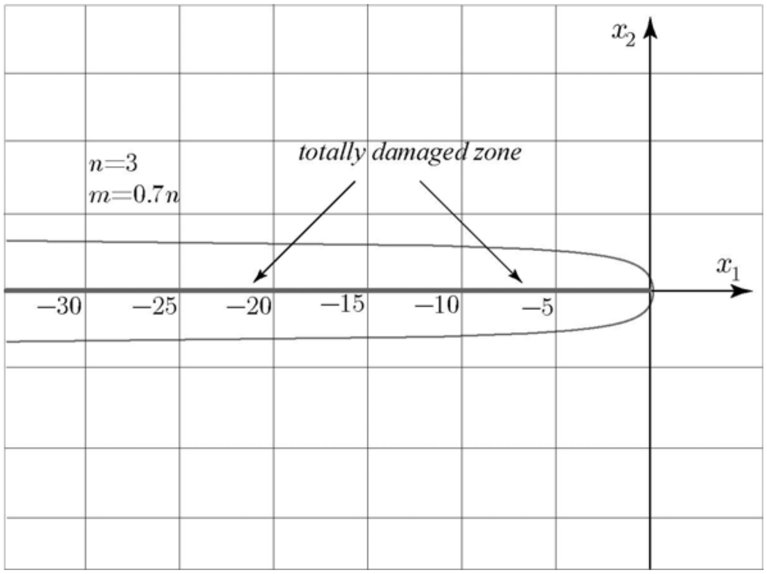

FIGURE 4. Geometry of the totally damaged zone adjacent to the crack surfaces. Mode I crack (plane strain conditions).

Conclusion

1) The new far field stress asymptotic behavior determining the geometry of the totally damaged zone enclosing the growing crack is found and analyzed. The results obtained are coincident with the eigenvalues for some values n given in Meng and Lee [8].

2) One can conclude that in the framework of the presented technique the unknown boundary of the totally damaged zone is completely described in a rather simple way by the only equation following from the two-term asymptotic expansion of the continuity parameter at large distance from the crack tip unlike some previous works where the contour of the damage field is a priori assigned, for example, by a semi-ellipse in front of the crack and a wake parallel to the crack plane behind the crack.

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