CRACK TIP STRESS FIELDS FOR ANISOTROPIC MATERIALS WITH CUBIC SYMMETRY

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Abstract

Fracture mechanics of linear elastic materials is generally based on the *K* field which has been derived for isotropic materials. Many applications require the use of advanced materials, which are often anisotropic and thus the isotropic elastic *K*-field is not applicable. In this work, a sharp crack lying in a homogenous, anisotropic material with cubic symmetry is studied and crack tip stress fields are presented. It is shown that the crack tip fields depend on material properties through the anisotropy factor, ρ. The stress fields are applicable for both plane stress and plane strain conditions, though the definition of ρ is different in each case. The theoretical *K* field obtained has been compared to results from finite element studies and excellent agreement has been obtained.

Introduction

The majority of solutions for crack tip fields address the problem of a crack within an isotropic material and are based on the isotropic *K*-field [1]. In this work, crack tip fields for materials with cubic symmetry only are examined. The motivation for the work arises from the increasing use of single crystal nickel alloys (which have elastic cubic symmetry) in gas turbine blades.

In [2], Sih *et al.* investigated the stress fields in the vicinity of a sharp crack lying within an anisotropic solid. Following the approach presented in [3] the elastic crack tip fields were obtained and it was shown that the square root crack tip singularity is present in the anisotropic case. More recently, stress fields for interface cracks in linear elastic anisotropic bimaterials have been examined in [4], [5]. Fourier transforms were used in [4], [5] to obtain the full crack tip fields of a finite crack lying along the interface between anisotropic elastic media. More recently, crack tip fields around kinked cracks in anisotropic elastic solids were obtained [6], [7]. In [6], the Stroh formalism is used to derive crack tip fields, in anisotropic elastic solids, including elastic *T*-stresses and other coefficients of the higher-order terms. Numerical results for various kink angles and mode mixities, were presented. The problem of an inclined crack in an orthotropic medium was studied in [8] and fully analytical solutions for crack tip fields in orthotropic materials have been obtained using complex potentials.

One of the drawbacks of these earlier works is that the precise form of the crack tip stress distributions has not been provided, so analytical studies of cracks in anisotropic materials cannot be based directly on these works. Here, our attention is directed to anisotropic materials with cubic symmetry. The approach adopted in [2] is used to determine the stress fields in the vicinity of a sharp crack in such a material. Solutions are provided for Mode I loading. The semi-analytical results are compared to those obtained from a finite element analysis of a crack in an infinite plate under tensile loading.

Crack Tip Stress Fields

The generalised Hooke's law for anisotropic materials can be expressed as:

$$\varepsilon_{ij} = S_{ijkl}\sigma_{kl}\,, \qquad\qquad \sigma_{ij} = C_{ijkl}\varepsilon_{kl}\,, \tag{1}$$

where ε_{ij} and σ_{ij} are the coefficients of the stress and strain tensors respectively and S_{iklj} and C_{ijkl} are the coefficients of the fourth order tensors (the compliance and stiffness matrix respectively) which contain the material constants for an elastic body. A material with cubic symmetry has four-fold rotational symmetry. For such a material there are three independent material properties and Eq. 1 has the following matrix form:

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{31} \\ 2\varepsilon_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{12} & 0 & 0 & 0 \\ S_{12} & S_{11} & S_{12} & 0 & 0 & 0 \\ S_{12} & S_{12} & S_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{44} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{bmatrix} \tag{2}$$

where

$$S_{11} = \frac{1}{E_{[001]}} \quad ; \qquad S_{12} = -\frac{\nu}{E_{[001]}} \quad ; \qquad S_{44} = \frac{1}{G_{[001]}} \tag{3}$$

Here, axes are designated as [100], [010] and [001] corresponding to x, y and z. In Eq. (3), $E_{[001]}$ represents the tensile modulus in the [001] direction for the cubic material, $G_{[001]}$ the shear modulus in the [001] direction and ν is the ratio between normal, ε_{11}, and transverse strain, ε_{22} (Poisson’s ratio).

The plane problem for an infinitely sharp crack is illustrated in Fig. 1. Assuming that stress gradients in the out of plane (z) direction are negligible, the out of plane shear stress, σ_{xz} and σ_{yz} are zero and the stresses depend only on x and y. For the plane stress/plane strain problem examined here, Eq. (2) can then be rewritten as,

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} s_{11} & s_{12} & 0 \\ s_{12} & s_{11} & 0 \\ 0 & 0 & s_{44} \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix}, \tag{4}$$

where $\gamma_{xy} = 2\varepsilon_{xy}$ and

$$s_{ij} = \begin{cases} S_{ij} & \text{plane stress} \\ S_{ij} - \dfrac{S_{i3}S_{j3}}{S_{33}} & \text{plane strain} \end{cases} \tag{5}$$

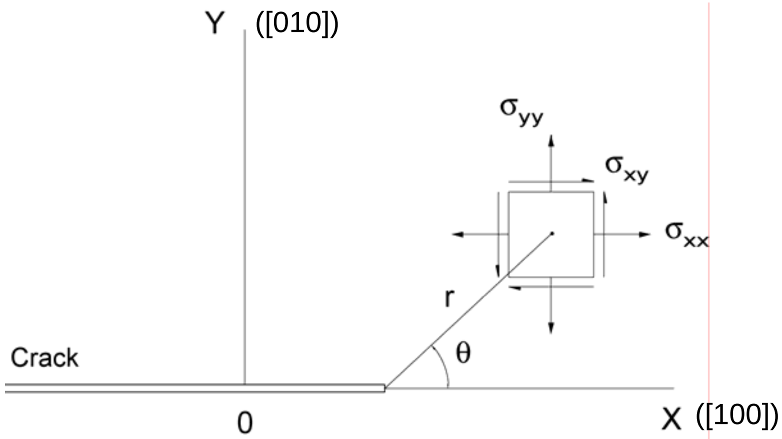

FIGURE 1. Plane crack problem.

The general anisotropic problem in two dimensions has been studied in [9]. Here, we specialise to the case of materials having cubic symmetry. As discussed in [10], the governing differential equation for the plane problem of a cubic material can be written in terms of a stress function, $U(x, y)$, such that,

$$s_{11}\frac{\partial^4 U}{\partial x^4} + (2s_{12} + s_{44})\frac{\partial^4 U}{\partial x^2 \partial y^2} + s_{11} = 0\,. \tag{6}$$

The characteristic equation of the differential equation, Eq. (6), is then

$$s_{11}\mu^4 + (2s_{12} + s_{44})\mu^2 + s_{11} = 0 \tag{7}$$

and the problem reduces to finding the roots of Eq. 7. Substituting the values of s_{ij} from Eq. 3 and leaving out the subscript [001] for simplicity, one obtains for a cubic material under plane stress conditions,

$$\mu^4 + \left[\frac{E}{G} - 2\nu\right]\mu^2 + 1 = 0. \tag{8}$$

By introducing a factor ρ [12],

$$\rho = \frac{2s_{12} + s_{66}}{2\sqrt{s_{11}s_{22}}} \tag{9}$$

and defining s_{ij} as in Eq. 5, Eq. 8 can be written for plane stress or plane strain as,

$$\mu^4 + (2\rho)\mu^2 + 1 = 0\,. \tag{10}$$

Equation 10 has four distinct complex roots, which may be written as,

$$u_1 = x_1 + iy_1, \qquad u_2 = x_2 + iy_2, \qquad u_3 = \bar{u}_1, \qquad u_4 = \bar{u}_2 \tag{11}$$

where $\bar{u}_i$ indicates the conjugate of u_i.

Having determined the roots of the characteristic equation, crack tip stresses may be determined for Mode I loading (following [9]) as,

$$\sigma_{xx} = \frac{K_I}{\sqrt{2\pi r}} \mathrm{Re}\left[\frac{u_1 u_2}{u_1 - u_2} \left(\frac{u_2}{\sqrt{\cos\theta + u_2 \sin\theta}} - \frac{u_1}{\sqrt{\cos\theta + u_1 \sin\theta}} \right) \right],$$

$$\sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}} \mathrm{Re}\left[\frac{1}{u_1 - u_2} \left(\frac{u_1}{\sqrt{\cos\theta + u_2 \sin\theta}} - \frac{u_2}{\sqrt{\cos\theta + u_1 \sin\theta}} \right) \right], \tag{12}$$

$$\sigma_{xy} = \frac{K_I}{\sqrt{2\pi r}} \mathrm{Re}\left[\frac{u_1 u_2}{u_1 - u_2} \left(\frac{1}{\sqrt{\cos\theta + u_1 \sin\theta}} - \frac{u_1}{\sqrt{\cos\theta + u_2 \sin\theta}} \right) \right].$$

In Eqs. 13, *Re* represents the real part of the complex number and K_I is the stress intensity factor under Mode I loading. It is seen in Eqs. 12 that a square root singularity is still present for the case of a cubic anisotropic elastic body. However, the angular functions in Eqs. 12 depend on the material properties through the characteristic equation, Eq. 10.

The angular functions in Eqs. 12 cannot be plotted directly as they depend implicitly on ρ through Eq. 10. Therefore, the characteristic equation, Eq. 10, is first solved analytically for a given value of ρ and the Real parts of the solution in Eq. 12 determined numerically (in this work the software package *Mathematica* [11] was used for this purpose). In this way, a semi-analytical solution for the crack tip fields is obtained. Note that the angular distribution is the same for plane stress and strain, though the value of ρ will depend on whether plane stress or plane strain conditions are assumed.

RESULTS

The angular distributions for Mode I loading are shown in Fig. 2 as a function of the crack tip angle θ. Finite element (FE) solutions are also provided in this figure, which will be discussed in the next section. Results are provided for ρ values varying from –0.5 to 1.0. It can be seen in Fig. 2(b) and (c) that the hoop and shear stress, $\sigma_{\theta\theta}$ and $\sigma_{r\theta}$, are relatively independent of the anisotropy parameter ρ, in contrast to σ_{rr} which is strongly dependent on ρ as shown in Fig. 2(a) (Tabulated solution for various values of ρ and for Mode I and Mode II are given in [10]).

For the isotropic case, corresponding to ρ = 1, Eq. 10 has equal roots and the solution obtained from Eqs. 10 and 12 oscillates. Therefore, the dotted line for ρ = 1 in Fig. 2 represents the isotropic Mode I *K*-field [1]. (Note that for ρ = 0.999 there was no oscillation of the semi-analytical solution and the result is indistinguishable from the isotropic *K*-field.)

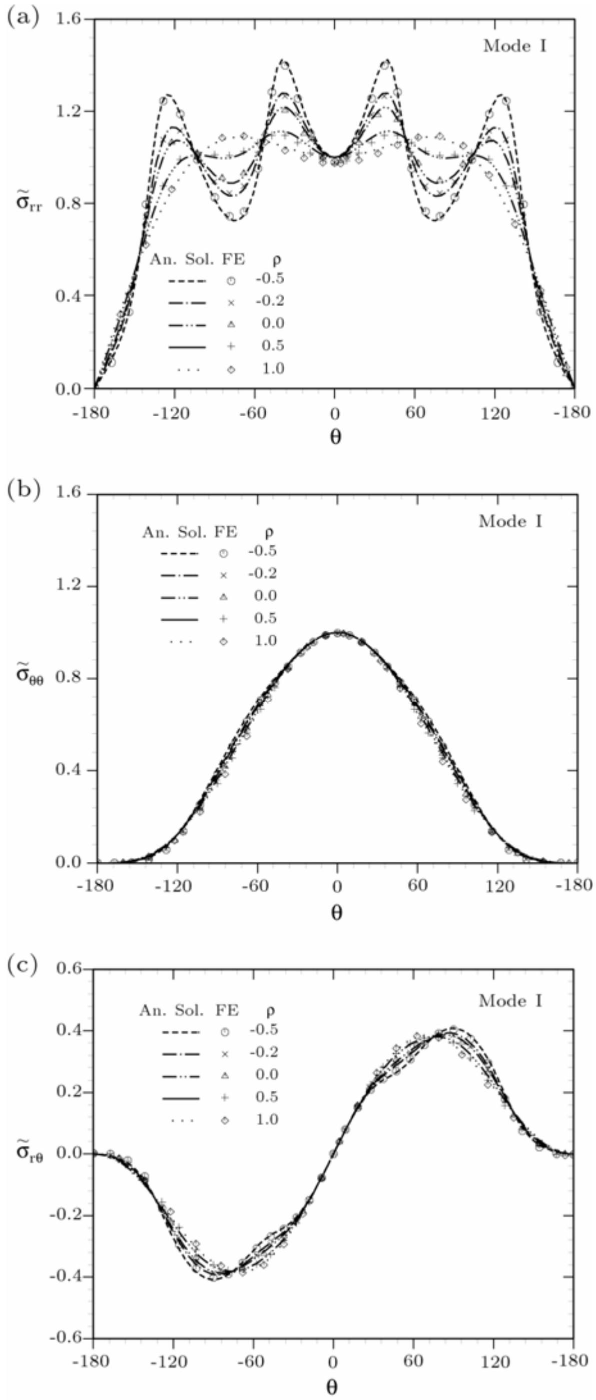

FIGURE 2. Analytical solution (lines) and finite element solution (symbols) for angular functions of a cubic material under mode I loading.

Using Eqs. 10 and 12, the angular variation of the stresses have been plotted for various values of ρ and compared with the results from finite element calculations. Some typical values of ρ are aluminium: ρ = 0.74, copper: 0.03, iron: 0.20 [12]. For the single crystal nickel alloys, CMSX4 and CM186, ρ = 0.1 [13] and 0.12 [14], respectively.

Finite Element Analysis

The semi-analytical crack tip fields obtained in the previous section are compared with finite element (FE) solutions. The FE calculations were conducted, using the finite element code ABAQUS [15], on a centre cracked plate loaded in tension (see Fig. 3). The crack length to specimen width ratio a/W = 0.1, which essentially corresponds to a crack in an infinite plate. The material was modelled as being linear elastic.

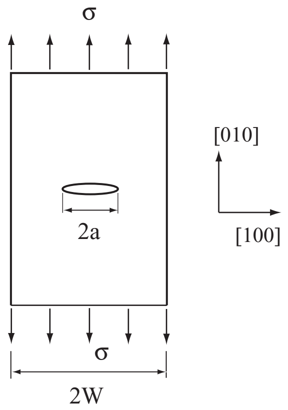

FIGURE 3. Loading configuration of a Mode I specimen

The FE mesh used for the analysis is shown in Fig. 4. A full mesh is employed to allow a range of mixed mode configurations to be examined in future work. In the figure three parts composing the mesh, are shown. Section (b) fits into the rectangular white area of (a) and section (c) fits into the white circles of (b). The current analysis was conducted under plane stress conditions using four node bilinear, reduced integration elements. The mesh used is composed of about 7000 elements, with the smallest size element being on the order of 1×10^{-5} of the crack length.

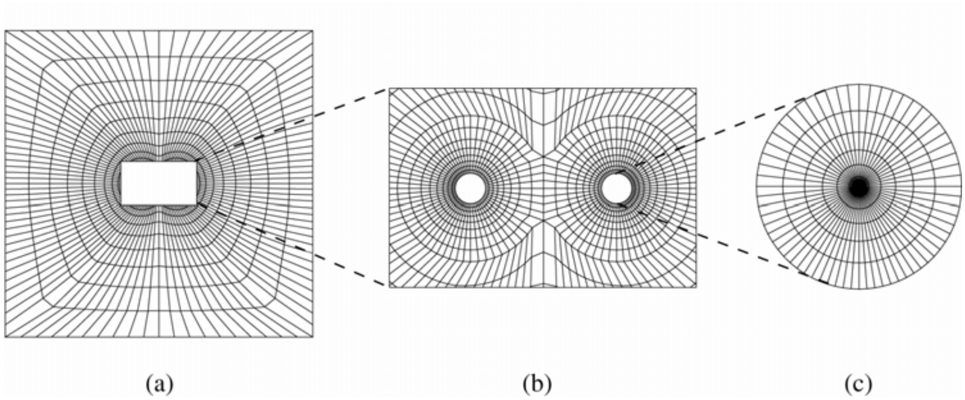

FIGURE 4. Mesh used for the FE analysis. Part (b) fits into the rectangular white area of (a) and part (c) fits into the white circles of part (b).

In Fig. 2, the comparison between the semi-analytical solution and the FE analysis is shown. The open symbols represent the FE solution and the lines are the semi-analytical solution. The FE stresses have been obtained at a distance $r/a = 2 \times 10^{-4}$, where the K-field is expected to be dominant. It can be seen that excellent agreement is obtained between the FE and the analytical solution.

Conclusions

In this paper the precise form of the crack tip stress distributions in a cubic anisotropic material have been obtained under plane stress/strain conditions. The solutions obtained are semi-analytical: the complex roots of the characteristic equation are first determined for a given value of the anisotropy parameter ρ, and the crack tip stress fields are then obtained as a function of these roots. The crack fields thus depend on the material properties only through the parameter ρ. It is found that the radial stress component σ_{rr} depends strongly on ρ whereas the other two in-plane components, $\sigma_{\theta\theta}$ and $\sigma_{r\theta}$ are almost independent of ρ. Finite element calculations have been conducted on a large plate containing a sharp crack and excellent agreement is obtained between the semi-analytical and FE results.

References

1. Irwin, G. R., *J. Appl. Mech.*, vol. **24**, 361–364, 1957
2. Sih, G.C., Paris, P.C. and Irwin, G.R., *Int. J. Fract. Mech.*, vol. **1**, 189–203, 1965
3. Lekhnitskii, S.G., *Theory of Elasticity of an Anisotropic Body*, Holden-Day, San Francisco, USA, 1963
4. Bassani, J.L. and Qu, J., *J. Mech. Phys. Solids*, vol. **37**, 435–453, 1989
5. Qu, J. and Bassani, J.L., *ibid*, 417–433, 1989
6. Yang, S. and Yuan, F.G., *Int. J. Solids Struct.*, vol. **37**, 6635–6682, 2000
7. Yang, S. and Yuan, F.G., *Int. J. Fracture*, vol. **101**, 291–319, 2000
8. Nobile, L. Piva, A. and Viola, E., *Eng. Fract. Mec.*, vol. 71, 529–546, 2004
9. Sih, G.C. and Liebowitz, H., In: *Fracture: An Advanced Treatise*, edited by H. Liebowitz, vol. **2**, Academic Press, New York, 1968, 67–190
10. Lempidaki, D.E. O'Dowd, N.P. and Busso, E.P., work in preparation, 2004
11. Wolfram Research, Inc. *Mathematica 4*, 1999.
12. Suo, Z., *J. Appl. Mech.*, vol. **57**, 627–634, 1990
13. Dennis, R.J. *Mechanistic Modelling of Deformation and Void Growth Behaviour in Superalloy Single Crystals.* PhD Thesis, Imperial College London, 2000
14. Regino, G.M. *A Multiscale Constititive Approach to Model the Mechanical Behaviour of Inhomogeneous Single Crystal Superalloys*, PhD Thesis, Imperial College London, work in preparation, 2004
15. Hibbitt, Karlsson and Sorensen Ltd. *ABAQUS*, *Version 6.1.* 2000