A COHESIVE ZONE MODEL WITH BARENBLATT SURFACE ENERGY : THEORETICAL CONSIDERATIONS AND NUMERICAL SIMULATIONS

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Abstract

By using a principle of least energy and a Barenblatt-type surface energy assumed as a smooth function of the jump of the displacements, we prove that it is possible to predict the onset of cracking by searching relative minima of the total energy of a body, the yield criterion depending on the choice of the energies. These types of brittle fracture models are then tested numerically by using cohesive elements.

1. Introduction

It is known, see Francfort and Marigo [1], that the linear fracture Mechanics based on Griffith's hypotheses relative to the surface energy (the surface energy associated to a surface of discontinuity is proportional to the area of this surface) and on the criterion formulated in terms of the critical energy release rate is (generally) unable to predict crack initiation in an initially sound elastic structure.

One of the remedies proposed by Francfort and Marigo was at first, while keeping Griffith's hypothesis on the surface energy, to replace the criterion of propagation by a principle of least energy (Griffith revisited). This principle, along with numeric adapted methods (see Bourdin *et al.* [2]), allows to widen the frame of application of Griffith's theory and is able to predict initiation and propagation of cracks following *a priori* non restricted space-time paths : multi-cracking, brutal propagation, branching, … . In counterpart the seek of **global** minima leads to non-admissible defects like spurious size effects or the inability of the material to sustain body forces. Moreover, considering only local minima is (almost) equivalent to Griffith's criterion and its incapacity to predict the initiation.

As a remedy, we propose to change the form of the surface energy by adopting Barenblatt's idea. In the spirit of what happens at an atomic scale when atomics bonds break we will assume that the surface energy depends on the value of the displacement jump through the crack. This approach corresponds to the cohesive zone models (CZM) frequently developed in the literature. With this change of the form of the energy and the seek of **local** minima we remedy to the main defects of Griffith's theories (original or revisited). It is at least what suggest analyses in dimension one made by Del Piero and Truskinovsky [4] or by Charlotte *et al.* [3] where it is shown that one of the necessary conditions of local minimum is that, in each sound point of the structure, the stress is lower than a critical stress corresponding to the derivative (with regard to the displacement jump) of the surface energy at zero. In other words the principle of local minimum applied in the case of a surface energy of Barenblatt's type leads to a yield criterion formulated in terms of the stress tensor.

We suggest here to generalize this result for three-dimensional structures. This is the object of the following section. We will then present the numerical implantation of this type of model by using cohesive elements. Finally this brittle fracture model will be tested numerically on a few configurations of fracture.

2. Surface energy and yield criterion

We consider three-dimensional homogenous isotropic brittle bodies. The energy E_s necessary to create a surface of discontinuity Γ of normal n and through which the displacement undergoes a discontinuity δ is, by following the idea of Barenblatt :

$$E_s = \int_\Gamma \varphi(\delta, n) d\Gamma$$

where φ is a concave function, vanishing if $\delta = 0$ and growing towards the tenacity G_c when the jump displacement δ tends to infinity. Moreover, for an isotropic material, the function surface energy φ has to satisfy the condition $\varphi(\boldsymbol{Q}\delta, \boldsymbol{Q}n) = \varphi(\delta, n)$ for any orthogonal matrix $\boldsymbol{Q}$. By decomposing the jump displacement in a normal and a tangential part : $\delta = \delta \cdot n\, n + \delta_t$, $\delta_t \cdot n = 0$, the surface energy can read as :

$$\varphi(\delta, n) = \phi\left(\delta \cdot n, \left\|\delta_t\right\|\right)$$

To prevent the interpenetration of the lips of the crack we impose $\delta \cdot n \geq 0$. Finally we suppose that $\phi(0,0) = 0$ and that the derivative of ϕ at $(0,0)$ is defined by :

$$\lim_{h \to 0+} \frac{1}{h} \phi(h\alpha, h\beta) = \psi(\alpha, \beta) \quad \text{with } \psi \text{ positively homogeneous of degree 1.}$$

Let us consider such a structure submitted to a given load. The state of this structure is characterized by the displacement field *u* and the stress field $\boldsymbol{\sigma}$ (cracks are identified with the surfaces of discontinuity of *u* and thus known once *u* is known). A necessary and sufficient condition so that the state of the structure is a locally stable equilibrium state is that it is a *local minimum of the energy*. In other words, there exists a neighbourhood (according to the chosen norm) of *u* such that the energy of the structure in this state is less than the energy of the structure in any other admissible field *v* in this neighbourhood.

The energy of the structure is constituted here by three terms : the elastic energy (defined on the sound zones of *v*), the surface energy (defined on the surfaces S_v of discontinuity of *v*) and the potential of the dead loads $f(v)$:

$$\mathrm{E}(v) = \int_{\Omega \setminus S_v} \mathbf{A}\boldsymbol{\varepsilon}(v) \cdot \boldsymbol{\varepsilon}(v)\, dx + \int_{S_v} \phi\left(\llbracket v \rrbracket \cdot n, \left\|\llbracket v_t \rrbracket\right\|\right) d\Gamma - f(v) \quad \text{with } \llbracket v \rrbracket \text{ : jump of } v$$

From a mathematical point of view, the most convenient space of displacement fields is the space of bounded variation which gives the norm used in the condition of local minimum. However, in this paper, we will limit our attention to piecewise smooth displacement fields. Thus the condition of local minimum for *u* reads as :

$$\exists r > 0 \quad \forall v : \left\|v - u\right\| \leq r \quad E(u) < E(v)$$

The **necessary** condition so that *u* is a local minimum, called *optimality condition of first order (OC1),* formally reads as $E'(u)(v) \geq 0$ for any admissible direction *v*. From classic arguments of calculus of variations, we deduce that the *OC1* are equivalent to the following **local conditions** :

(a) Equilibrium equations : $div\,\boldsymbol{\sigma} + f = 0 \quad in \quad \Omega \setminus S_u$

(b) Boundary condition : $\boldsymbol{\sigma} n = F \quad on \quad \partial_F \Omega$

(c) Jump condition : $\boldsymbol{\sigma} n = \phi_{,1}\left(\delta \cdot n, \|\delta_t\|\right) n + \phi_{,2}\left(\delta \cdot n, \|\delta_t\|\right) \dfrac{\delta_t}{\|\delta_t\|} \quad on \quad S_u$

(d) Yield criterion : $\boldsymbol{\sigma} n \cdot \nu \leq \psi\left(n \cdot \nu\,,\ \|\nu - \nu \cdot n\, n\|\right)\,, \quad \forall n, \forall \nu : \|n\| = \|\nu\| = 1 \quad in\ \ \Omega \setminus S_u$

with $\boldsymbol{\sigma} = \mathbf{A}\boldsymbol{\varepsilon}(u)$ and $\delta = [\![u]\!] = u^+ - u^-$.

Depending on the properties of the derivative of ϕ *at* $(0,0)$ we will find different types of yield criteria.

- *If* ϕ *is differentiable at* $(0,0)$: ψ is linear and we have $\psi(\alpha, \beta) = \sigma_c \alpha + \tau_c \beta$, σ_c and τ_c being characteristic stresses of the material. The yield criterion *(d)* becomes :

$$\boldsymbol{\sigma} n \cdot \nu \leq \sigma_c\, n \cdot \nu + \tau_c \|\nu - \nu \cdot n\, n\|\,, \quad \forall n, \forall \nu : \|n\| = \|\nu\| = 1 \quad in\ \ \Omega \setminus S_u$$

which can be written in term of principal stresses $\{\sigma_i\}_{i=1,3}$:

$$\boxed{\sup_{1 \leq i \leq 3} \sigma_i \leq \sigma_c \quad and \quad \sup_{1 \leq i, j \leq 3} |\sigma_i - \sigma_j| \leq \tau_c}.$$

It's a criterion on both *maximal traction* and *maximal shear* :

- *If* ϕ *is not differentiable at* $(0,0)$: When ϕ admits only directional derivatives at $(0,0)$, ψ is no more linear. By using Legendre transform, it appears that the criterion *(d)* is of intrinsic curve type, the maximal shear stress decreasing monotonically with the normal stress. It depends only on the extreme principal stresses σ_1 and σ_3 and can be written :

$$\boxed{|\sigma_1 - \sigma_3| + 2\psi_*\left(\frac{\sigma_1 + \sigma_3}{2}\right) \leq 0} \quad \text{with} \quad \psi_*(s) = \sup_{\omega \in [0, \pi/2]} \left\{ s\cos\omega - \psi(\cos\omega, \sin\omega) \right\}$$

The difference in comparison to Griffith's theory lives in the conditions *(c)* and *(d)*. For Griffith's theory, the first one would read $\boldsymbol{\sigma} n = 0$, which implies the absence of interaction between the lips of cracks, while the second one would be : $G \leq G_C$ the famous Griffith criterion which is global and no more local. We see that with a Barenblatt's surface energy type, the lips of the cracks interact, which correspond to the notion of cohesive forces.

Remarks: The *OC1* are only necessary conditions for *u* to be a local minimum. It is necessary to add to them second order conditions which are generally global ones (see [3] or [4]). For more details on this part see Laverne and Marigo [5].

3. Numerical Model

One of the main difficulties of numerical implantation of such models lives in the necessity to suppose *a priori* any surface of discontinuity in the structure. This difficulty had been by-passed in the revisited Griffith theory by making an "elliptic regularization" of the energy which allowed to work with regular fields (see [2]), the discontinuities being replaced by strong gradients. Such a process of regularization is not available yet for Barenblatt's surface energies. We have therefore opted for an implantation (in two dimensions) by cohesive elements with internal discontinuities. In other words we suppose the crack path is known *a priori*.

3.1 Choice of the surface energy

We choose a density of surface energy depending only on the norm of the jump displacement and, on this fact, independent of the orientation *n* of the discontinuity line :

$$\varphi(n,\delta) = G_c\left(1-\exp\left(-\frac{\sigma_c}{G_c}\|\delta\|\right)\right)$$

We note that it's a concave function of the norm of the jump displacement, starting from zero and progressively growing to the tenacity G_c (see [3]).

3.2 Choice of the cohesive element

The element is a quadrangle with an internal discontinuity noted [AB] which is located at the centre of the sides [1-4] and [2-3] (see Fig. 1). It allows to define a local mark in the element : *n* and *t* are respectively a normal and a tangent unit vector to the discontinuity line. The main idea consists in considering the jump of displacement in the element, denoted $\delta = (\delta_n, \delta_t)$, as a *constant* "internal" variable. Moreover we consider that the normal jump δ_n cannot be strictly negative to take into account the condition of non-interpenetration of the crack lips.

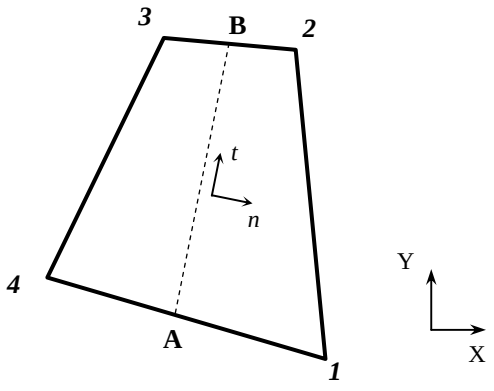


FIGURE 1. Geometry of the cohesive element.

3.3 Numerical solving

By using a principle of least energy the solution of the problem consists in finding the displacement field as well as the jumps in each cohesive element.

$$\textbf{Minimisation problem} : \text{find } \left(U^*,\delta^*\right) = \arg\min_{U,\delta}\left(E\left(U,\delta\right)\right)$$

Numerically, the seek for a local minimum of the total energy at a given level of loading will be made in two steps : successive minimization with respect to the jump then with respect to the displacement.

Remark : in this part we will detail only the computation of the jump in a cohesive element for the *onset of cracking*. The calculation of the jump for *propagation* is similar. We take into account the irreversibility of cracking thanks to a threshold variable.

• *First step : Determination of* δ *for a fixed* U. Thanks to the choice of δ constant by element, a local analysis is sufficient to determine δ in every element according to the displacement. Indeed, for given U, δ will be a minimizer of the element total energy. Supposing that the element is free of body forces, its energy reads as (with matrix notations) :

$$E(U,\delta) = \frac{1}{2}\int_V \left(\boldsymbol{B}U - \boldsymbol{D}\delta\right)^T \mathbf{A}\left(\boldsymbol{B}U - \boldsymbol{D}\delta\right) dV + LG_c\left(1-\exp\left(-\frac{\sigma_c}{G_c}\|\delta\|\right)\right)$$

the first term corresponds to the elastic energy (the strain $\boldsymbol{\varepsilon}$ read as $\boldsymbol{\varepsilon} = \boldsymbol{B}U - \boldsymbol{D}\delta$ and the stress $\boldsymbol{\sigma} = \mathbf{A}\boldsymbol{\varepsilon}$) and the second one to the surface energy (L is the length of the discontinuity line AB on Fig. 1). E is a strictly convex function of δ as long as the size of the element is small enough, which fixes the maximal mesh size. There is then a unique minimum. We obtain (by noting $\langle x\rangle^+ = Max\{x,0\}$) :

If $\left(\left\langle n^T\boldsymbol{D}^T\mathbf{A}\boldsymbol{B}U\right\rangle^+\right)^2 + \left(t^T\boldsymbol{D}^T\mathbf{A}\boldsymbol{B}U\right)^2 \le L^2{\sigma_c}^2$, then $\delta = 0$. It corresponds to the onset criterion *(d)* in part 2. Otherwise the jump reads as :

$$\delta = a\left(\frac{t^T\boldsymbol{D}^T\mathbf{A}\boldsymbol{B}U}{t^T\boldsymbol{D}^T\mathbf{A}\boldsymbol{D}t\ a + L\sigma_c\exp\left(-\frac{\sigma_c}{G_c}a\right)}t + \frac{\left\langle n^T\boldsymbol{D}^T\mathbf{A}\boldsymbol{B}U\right\rangle^+}{n^T\boldsymbol{D}^T\mathbf{A}\boldsymbol{D}n\ a + L\sigma_c\exp\left(-\frac{\sigma_c}{G_c}a\right)}n\right)$$

with $a > 0$ solution of the following non linear scalar equation :

$$1 = \left(\frac{t^T\boldsymbol{D}^T\mathbf{A}\boldsymbol{B}U}{t^T\boldsymbol{D}^T\mathbf{A}\boldsymbol{D}t\ a + L\sigma_c\exp\left(-\frac{\sigma_c}{G_c}a\right)}\right)^2 + \left(\frac{\left\langle n^T\boldsymbol{D}^T\mathbf{A}\boldsymbol{B}U\right\rangle^+}{n^T\boldsymbol{D}^T\mathbf{A}\boldsymbol{D}n\ a + L\sigma_c\exp\left(-\frac{\sigma_c}{G_c}a\right)}\right)^2$$

• *Second step : Determination of U.* Once δ is determined in every cohesive element according to the displacement, we solve the discrete equilibrium equation (equivalent to *(a)* and *(b)* in part 2) $\boldsymbol{K}U = F$ where F, the vector forces, must be updated to take into account stresses associated to δ in cohesive elements.

• *Iterations :* For a given load level, the two steps are iterated : from an initial vector U^0 we build, until convergence, a sequence $\left(\delta^i, U^i\right)$ with a global Newton-Raphson algorithm.

4. Simulations

4.1 Validation test

The aim of this test case is to validate the numerical model by confronting it with an analytical solution. Let us consider a crack along which the jump displacement is not constant. The idea is then, in addition to a good prediction of the jump, to notice a decrease of the numerical error when we refine the mesh.

We consider a two-dimensional rectangular elastic plate Ω and Γ one of its sides. The Airy stress function $\Phi(x,y)$, governed by the equation : $\Delta\Delta\Phi = 0$ on Ω, when body forces vanish, generates stresses that satisfy the equations of equilibrium and compatibility (see Fung [8]). In this case σ_{xx}, σ_{yy} and σ_{xy} are derived from $\Phi(x,y)$ according to the following equations :

$$\sigma_{xx} = \frac{\partial^2 \Phi}{\partial y^2} \quad , \quad \sigma_{yy} = \frac{\partial^2 \Phi}{\partial x^2} \quad \textit{and} \quad \sigma_{xy} = -\frac{\partial^2 \Phi}{\partial x \partial y}$$

Let's chose a biharmonic function $\Phi(x,y)$ such that the stress tensor reads as :

$$\begin{cases} \sigma_{xx} = \alpha x + \beta y + \gamma \\ \sigma_{yy} = 0 \\ \sigma_{xy} = \alpha y + \eta \end{cases} \qquad \text{with } \alpha, \beta, \gamma \text{ and } \eta \text{ arbitrary constants.}$$

For a Young modulus : $E = 1$ and a Poisson's ratio : $\nu = 0$ we deduce the displacement field in the plate (and in particular the boundary conditions on $\partial\Omega$) :

$$\begin{aligned} u(x,y) &= \alpha \left(\frac{x^2}{2} + y^2 \right) + x(\beta y + \gamma) \\ v(x,y) &= -\beta \frac{x^2}{2} + 2\eta x \end{aligned} \tag{1}$$

Furthermore, knowing the vector $\boldsymbol{\sigma}.n$ on Γ, we can deduce the jump S on Γ, corresponding to such a force, by inverting Barenblatt's law. The new prescribed displacement on Γ, generating such a jump, thus reads as : $U_\Gamma = U_\Gamma^0 - S$ (with U_Γ^0 the prescribed displacement on Γ given by (1)). So we build an analytical solution of the elastic plate containing a crack along which the jump displacement is not constant.

Finally, for the numerical simulation, we put cohesive elements along Γ, we apply a displacement U_Γ on Γ and a displacement given by (1) on $\partial\Omega \setminus \Gamma$. Let us note S^{num} the numerical solution for the jump. We notice that the numerical error $\left\| S - S^{num} \right\|_{L^2(\Gamma)}$ decreases when we refine the mesh (see Fig. 2).

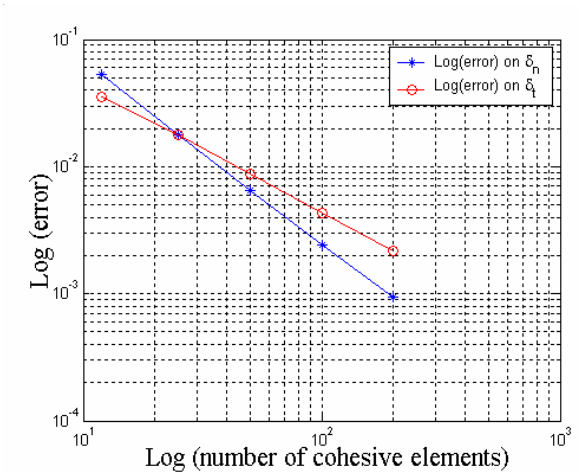


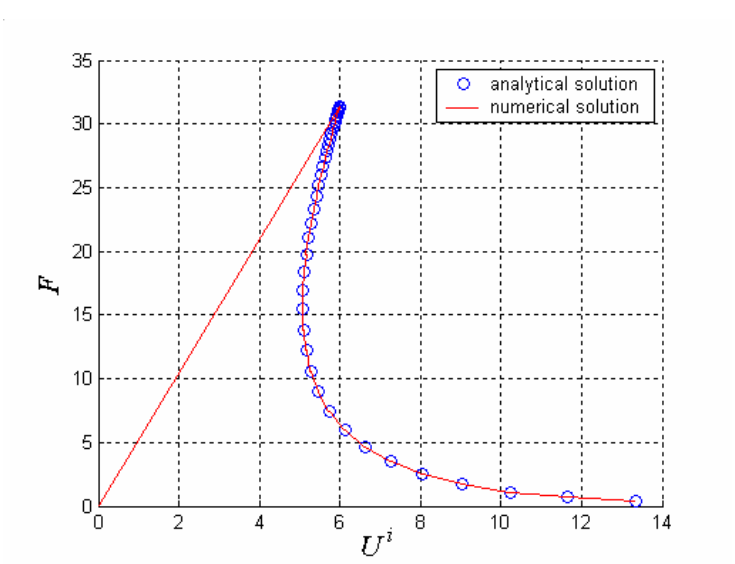
FIGURE 2. Error *vs.* number of cohesive elements (logarithmic scale).

4.2 The pull-out problem

This study concerns a cylindrical beam composed of two parts : a matrix with cross circular section reinforced by a fibre immersed on its centre. The matrix is assumed to be linear homogeneous isotropic material with an elastic behaviour while the fibre is taken rigid. The beam is clamped on its lateral surface, while a longitudinal displacement U^i is prescribed at the top of the fibre. Cohesive elements are located everywhere in the matrix so that we authorize only longitudinal debonding (see Fig. 3).

Moreover a technique of path-following of the load was developed to take into account a possible brutal opening of the cracks see Badel and Lorentz [7]. It allows to follow the unstable branches of the global response of the structure.

The numerical simulation leads to a debonding at the interface Matrix / Fibre on all the height of the beam. The results are validated by comparing the numeric curve of the global response with the analytical solution (see Fig. 4).



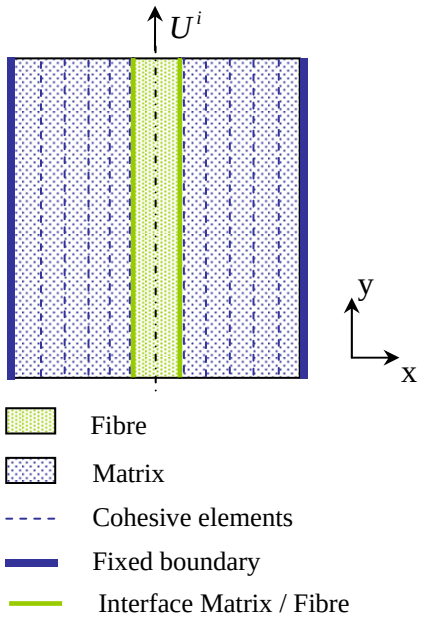


FIGURE 4. Global response of the beam. Force *vs.* prescribed displacement.

FIGURE 3. Geometry of the beam.

4.3 Crack propagation through a perforated plate

This study concerns the simulation of crack propagation through a perforated plate which is submitted to a prescribed force F_i applied on both ends, see Fig. 5. By postulating *a priori* the potential crack paths, we put cohesive elements along the symmetry line A-A' right and left from the hole. Global response and displacement field are given in Figs. 6-7.

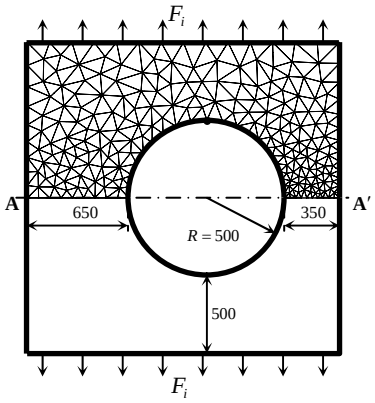


FIGURE 5. Geometry of the plate and load conditions (dimension in mm).

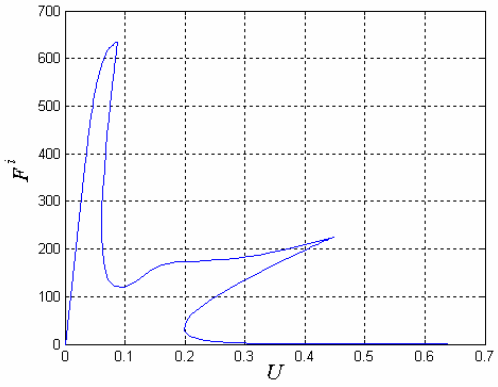


FIGURE 6. Global response of the plate. Prescribed force *vs* vertical displacement at the top of the hole.

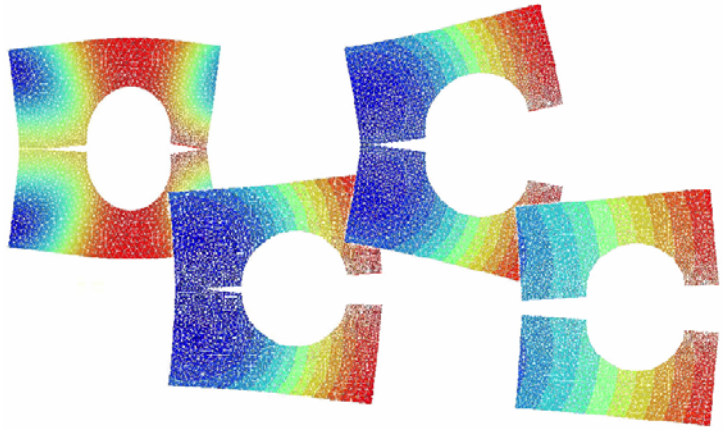


FIGURE 7. Evolution of the displacement field.

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