Modification of Dugdale model to include the work hardening and in and out of plane constraints

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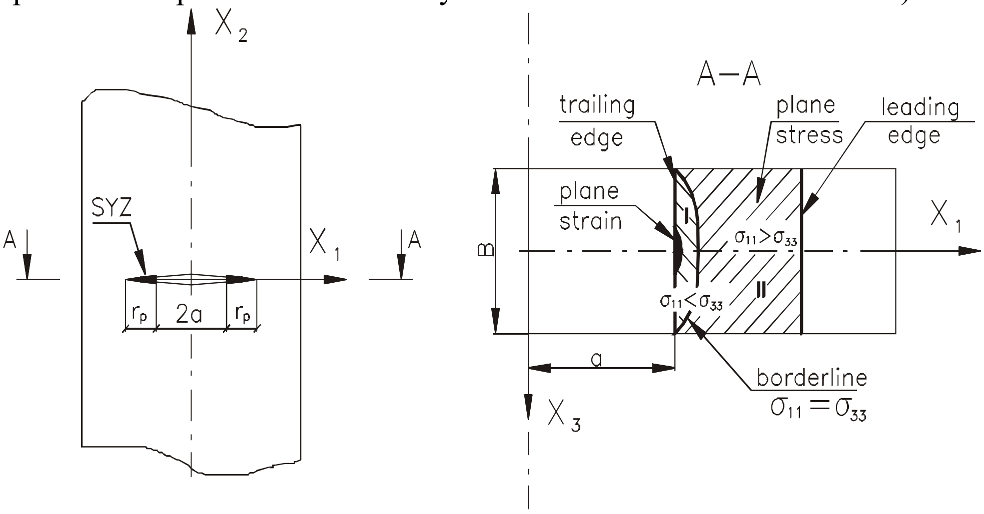
ABSTRACT: The paper is an extension of the author's earlier publication in which Dugdale strip yield zone (SYZ) model was generalized from thin specimens (modeled as dominated by plane stress condition) to specimens of arbitrary thickness. Now the model has been extended to include the in – plane constraint as well as the effect of the strain hardening of the material within SYZ.

INTRODUCTION

Tri-dimensional numerical stress analyses (e.g. [1-3]) in front of the crack within elastic-plastic materials show that plane strain dominates over the region close to the crack tip and midsection of specimen (see Fig.1b). In turn, plane stress is reached at certain distance from the crack tip. This distance depends on the x_3 coordinate. Fig.1 (along the midsection of the specimen the plane stress is usually reached at the distance $r=0.5B \div B$).



a) **b)**

Fig.1 The scheme of the 2-D strip yield zone (SYZ)

Two-dimensional finite element (FE) analyses using finite strains (e.g. [4]) show that the stress components σ_{11} and σ_{22} reach the maximum values, which depend on Ramberg-Osgood (R-O) work hardening power exponent n, at distance from the crack tip close to $2\delta_T$, where δ_T is crack tip opening displacement. From the location of maximum stress to the crack front both stress components decrease down to zero and σ_0 value respectively (σ_0 is the yield stress). These analyses show that the stress distribution in front of the crack depends not only on the plastic properties of material but also on the *in*- and *out-of-plane* constraints.

The purpose of this paper is to show that the well known Dugdale model [5], for Mode I, can be modified to include both *in*- and *out-of-plane* constraints as well as the work hardening of material. It may be achieved by requiring that the yield criteria should be satisfied within the SYZ.

IN- AND *OUT–OF–PLANE* CONSTRAINTS IN DUGDALE MODEL.

In the paper [6] author utilized the tri-axial stress constraint (TASC) factor, *Tz* introduced and defined by Guo [7] to extend the classic Dugdale model [5] to arbitrary specimen thickness. Original Dugdale model was introduced for *plane stress situation*, it means for thin specimens, and for *elastic – perfectly plastic materials*. SYZ models plastic domain in front of the crack. According to the original Dugdale model [5], only the Tresca yield condition is satisfied within this zone. For plane stress the smallest normal stress component is $\sigma_{33} = 0$ (notation is due to the coordinate system shown in Fig.(1)). The largest normal stress component is $\sigma_{22}=\sigma_0$. Tresca yield criterion does not take into account the σ_{11} stress component for plane stress situation.

For plane stress case Huber-Mises-Hencky (HMH) yield criterion requires that within SYZ $\sigma_{22} = \sigma_0 / \left(\alpha^2 - \alpha + 1\right)^{1/2}$, where $\alpha=\sigma_{11}/\sigma_{22}$. Usually, $\alpha=\alpha(x_1)$ and $0 \le \alpha < 1$. Thus, in most cases $\sigma_{22} > \sigma_0$ and σ_{22} must change along SYZ to satisfy this criterion.

When the plane strain is assumed neither HMH nor Tresca criterion is satisfied if classic Dugdale model is considered [6]. The stress analysis within the SYZ for perfectly plastic material showed that if the yield criteria are satisfied the stress component σ_{22} rise considerably above the yield stress [6]. The stress rise was due to the geometrical, *out-of-plane* constraint only.

In tri-dimensional case Tresca yield condition requires proper selection of the greatest and smallest normal stress component. The largest one, is σ_{22}; the smallest one may be either σ_{11} or σ_{33}. Analysis of the inequalities $\sigma_{11}<\sigma_{33}$ or $\sigma_{33}<\sigma_{11}$, utilizing definition of the TASC factor: $Tz=\sigma_{33}/(\sigma_{11}+\sigma_{22})$ (Tz may change from 0 (plane stress) to 0.5 (plane strain)), as well as the value of the ratio $\alpha=\sigma_{11}/\sigma_{22}$ (α is greater than zero and usually smaller than one) leads to the conclusions:

- $\sigma_{11}<\sigma_{33}$ within domain I (Fig.1). This inequality can not be satisfied in domain dominated by plane stress since it would require $\alpha<0$ for $Tz=0$.
- $\sigma_{33}<\sigma_{11}$ within domain II. This inequality can not be satisfied in domain dominated by plane strain since it would require $\alpha>1$ for $Tz\to 0$.
- $\sigma_{33}=\sigma_{11}$ along the *borderline* and the following conditions are satisfied:

$$Tz = \frac{\alpha}{1+\alpha} \qquad \text{or} \qquad \alpha = \frac{Tz}{1-Tz} \tag{1}$$

$$\sigma_{22} = \sigma_0 /(1-\alpha) \tag{2}$$

Within the domain I σ_{22} changes according to Eq (2) and α should satisfy Eq(1) at the borderline, it means at the line where stresses reach the maximum values. At the crack tip α should be equal to zero to satisfy the numerical results referred to in the Introduction.

Within the domain II σ_{22} changes according to the formula:

$$\sigma_{22} = \sigma_0 \left[1 - Tz(1+\alpha)\right]^{-1} \tag{3}$$

It can be easily shown that for small scale yielding (SSY) the ratio α at the leading edge (Fig.1) can be expressed by the relation:

$$\alpha(r = r_p) = 1 + \frac{T}{\sigma_0} \tag{4}$$

where T is the second term of the asymptotic expansion of the stress field in front of the crack in linear elastic material. For large scale yielding the similar relation can be written in terms of the Q–stress [4]. Thus, the *in-*

plane constraints are included into model through the T- or Q-stresses computed numerically.

Qualitative picture of the stress distribution within the SYZ according to the HMH hypothesis should not be much different from those obtained using the Tresca criterion. Thus, also for the HMH hypothesis, the borderline exists. At this line $\sigma_{11} = \sigma_{33}$ and the HMH hypothesis reduces to $\sigma_{22} = \sigma_0 /(1-\alpha)$ (the same relation as for Tresca criterion). The stress distribution within the SYZ, which satisfy the HMH criterion can be determined by the following formula [6] :

$$\sigma_{22} = \sigma_0 \left[(1+\alpha)^2 (Tz-1)Tz + \alpha^2 - \alpha + 1 \right]^{-1/2} \tag{5}$$

Functions $\alpha(\beta)$ and $Tz(\beta,\gamma,n)$ should be known (where $\beta = r/r_p$, $\gamma = r_p/B$, r_p is the SYZ length) to compute stress distribution within SYZ according to Eqs (2), (3) and (5),

The $\alpha(\beta)$ function can be approximated by a piece-wise linear function in between three points: at the leading edge of the SYZ (Eq. 2): $\alpha_2 = \alpha(r=r_p)$, at the borderline: $\alpha_1 = \alpha(\beta=\beta_b)$ (Eq. (4)) and at the trailing edge: $\alpha(r=0)$ where β_b denotes the location of the borderline along SYZ.

$$\alpha(\beta) = [\alpha_1(1-\beta) + \beta\alpha_2] H(\beta - \beta_b) + \frac{\alpha(\beta_b)}{\beta_b} \beta H(\beta_b - \beta), \tag{6}$$

where $H(-)$ is Heaviside function.
The TASC factor, Tz can be computed from:

$$Tz = \nu_{ep} F(\beta,\gamma) \left[1 - \left| \frac{2x_3}{B} \right|^{g(\beta,\gamma)} \right]^2, \tag{7}$$

where $\nu_{ep}(\nu,n)$ and $F(\beta,\gamma)$ are defined in [6,7].
Location of the borderline can be determined using Eqs (1), (2) and (7) for given or assumed maximum value of σ_{22} within the SYZ.

DUGDALE MODEL AND THE STRAIN HARDENING EFFECT.

It was shown in [6] that within the SYZ the σ_{22} stress rises above the yield stress by a considerable amount. The stress rise was due to the geometrical constraints only. Numerical computations performed for finite strains (e.g. [4]) show that the stress elevation is lower for perfectly plastic materials than for strain hardening ones. Thus, adaptation of the Dugdale model to real materials requires its modification to include work hardening in addition to the *in-* (α function) and *out* (*Tz* function)- *of - plane* constraints. To include the work hardening properties of material, the constitutive equation should be adopted. It requires proper definition of strains, which is not so obvious for the SYZ. We propose this definition in the form:

$$\varepsilon_{22} = \varphi \frac{\delta}{\delta_T} \tag{8}$$

where: $\delta(r)$ defines the opening of the crack faces within the SYZ, φ is scaling factor, which will be defined later. The simplified definition of $\delta(r)$, adopted in the paper is as follows:

$$\delta(r) = \delta_T \varphi (1-\beta)^{4s}, \tag{9}$$

where $s=\sigma_{ext}/\sigma_o$ and σ_{ext} is the external traction. Accuracy of this approximation is high for moderate loading: for $s=0.5$ the maximum error with respect to the exact solution [8] is less than 3 per cent. For $s=0.95$ the error does not exceed 8 percent. Above approximation is strictly correct for uniform stress distribution within the SYZ. Now the one – dimensional form of the R-O stress – strain constitutive relation can be adopted. For perfectly plastic material within the SYZ the yield stress can be written in the form:

$$\sigma_y = \sigma_0 + \sigma_0 \alpha^* \varepsilon^{1/n} = \sigma_0 \{1 + \alpha^* \left[\varphi (1-\beta)^{4s}\right]^{1/n}\} \tag{10}$$

where a^* (R-O coefficient) is assumed to be equal to 1. The scaling coefficient φ should be selected properly to satisfy the following requirements: 1) When $\beta=1$ the yielding stress, σ_y should be equal to σ_0. 2) When $n \to \infty$ the yielding stress should also approach the σ_0 value. 3) The yielding stress should reach the maximum value at the borderline since the opening stress decreases between the borderline and the trailing edge. All these requirements are satisfied if φ is defined as:

$$\varphi = \frac{\left[\left(\frac{\xi_n}{\xi_\infty}\right)^{\omega} - 1\right]^{n}}{(1-\beta_b)^{4s}} \tag{11}$$

where $\xi_n = (\sigma_{22})_{\max}/\sigma_0$ for R-O material characterized by power exponent n, $\xi_\infty = (\sigma_{22})_{\max}/\sigma_0$ for perfectly plastic materials. $(\sigma_{22})_{\max}$ denotes the value of the opening stresses at the borderline. The exponent ω is unknown quantity. It is expected (and it is assumed in this paper) that $\omega \cong 1$.For high constraint (e.g. double edge notched specimen) ξ_∞ = 3. In principle this value should be computed numerically as well as the value of ξ_n.

RESULTS OF COMPUTATIONS

All formulas presented in the two last chapters can be used to compute:

1. The stress distribution within the SYZ both for perfectly plastic materials and work hardening materials for arbitrary specimen thickness
2. The value of the average, uniform stresses distribution within the SYZ for arbitrary specimen thickness, for perfectly plastic materials and work hardening materials.

Results of the computations presented in this chapter were obtained using Mathcad 2001. Numerically obtained results were also approximated by simple algebraic formulas to allow for their applications to various practical Fracture Mechanics problems.

Computations should be made according to the following scheme:

- Using Eq. (2) the value of $\alpha(\beta=\beta_b)$ can be computed replacing σ_{22} with $(\sigma_{22})_{max}$, which should be known or assumed. For perfectly plastic material and highly constrained geometries this value is close to 3. For less constrained geometries it is smaller than 3. Such a value can be either adopted from the literature or obtained from FE computations. When the work hardening effect is taken into account one should also replace the σ_0 by σ_y computed from Eq. (10) for $\beta=\beta_b$.
- The value of $\alpha(\beta=\beta_b)$ should be used in Eq.(1) in order to find the value of $Tz(\beta=\beta_b)$ at the borderline.
- Utilizing value $Tz(\beta=\beta_b)$ along with Eq.(7) the location of the borderline, β_b should be determined.
- $\alpha(\beta)$ function should be computed introducing $\alpha(\beta=0)=0$, $\alpha_2=\alpha(\beta=1)$, computed from Eq.(2) and $\alpha_1=\alpha(\beta=\beta_b)$ into Eq.(4).

- Introducing Eqs (6) and (7) into Eqs (2) or (3) and (5) the stress distribution within the SYZ can be determined for Tresca or HMH yield conditions. When the work hardening is taken into account σ_0 should be replaced by σ_y (Eqs (10) and (11)).
- The average value of the SYZ stresses, σ_{SYZ} should be computed.

Below an example results are shown for: $\xi_\infty=3$, $\xi_n=5$, $\sigma_0=1000MPa$, $s=0.5$, $T=0$ for variety of specimen thickness ($\gamma=r_p/B$). The results will be compared with those for perfectly plastic materials.

The yield stress distributions, σ_y computed according to Eqs (10) and (11) are shown in Fig. (2). They have been computed for wide range of R-O exponents n. The results are practically independent of ξ_∞ and ξ_n for given ξ_n/ξ_∞ ratio. The yield stress distributions were used to compute the stress distributions within the SYZ using Eqs (5), (2) and (3) for the HMH and Tresca yield criteria respectively. The results are shown in Fig. (3) and they represent average, through the thickness values. One may notice the maximum of the opening stresses in front of the trailing edge. The work hardening exponent (at fixed ξ_n and ξ_∞) does not influence this distribution close to the trailing edge. Differences are noticeable close to the leading edge of the SYZ.

The main reason for the computations presented in this paper was to provide simple formula for the average stress distribution within the SYZ, $\sigma_{SYZ.}$ Such an approximation would preserve the simplicity of the Dugdale model application to variety of the Fracture Mechanics problems allowing however, for certain generalization of the model. The general formula to compute σ_{SYZ} is as follows:

$$\sigma^h_{SYZ}(\gamma)\Big|_{1<n<\infty} = \sigma^h_{SYZ}(\gamma)\Big|_{n=\infty}\left\{1+\left[\frac{\left(\frac{\xi_n}{\xi_\infty}-1\right)}{\left(1-\beta_b\right)^{2/n}}\right]\int_0^1 \frac{\sigma^h_{22}(\beta,\gamma)\big|_n}{\sigma^h_{22}(\beta,\gamma)\big|_{n=\infty}}(1-\beta)^{2/n}\,d\beta\right\} \quad (12)$$

where the superscript h can be replaced by “HMH” or “Tresca” depending on the hypothesis used in the analysis. Exact numerical results following from evaluation of the Eq. (12) are shown in Fig.(4) for different R-O exponents n. Eq. (12) can be simplified without noticeable lost of accuracy to the following formula:

$$\sigma_{SYZ}^{\omega}(\gamma)\Big|_{1<n<\infty} = \sigma_{SYZ}^{\omega}(\gamma)\Big|_{n=\infty}\left[1+\left(\frac{\xi_n}{\xi_\infty}-1\right)\Psi(n)\right] \tag{13}$$

where function *Ψ(n)* is approximation of the integral in Eq.(12) and it has a form:

$$\Psi(n) = \frac{1}{a+\dfrac{b}{n}} \tag{14}$$

where a=1, b=1.773 for $\xi_n=5$ and $\xi_\infty=3$. For other values of ξ_n/ξ_∞ ratio coefficients will be different. Function $\sigma_{SYZ}^{\omega}(\gamma)\Big|_{n=\infty}$ was presented in [6].

$$\sigma_{SYZ}^{\omega}(\gamma)\Big|_{n=\infty} = \sigma_0\left[e+\frac{f}{(\gamma/c)^d}\right] \tag{15}$$

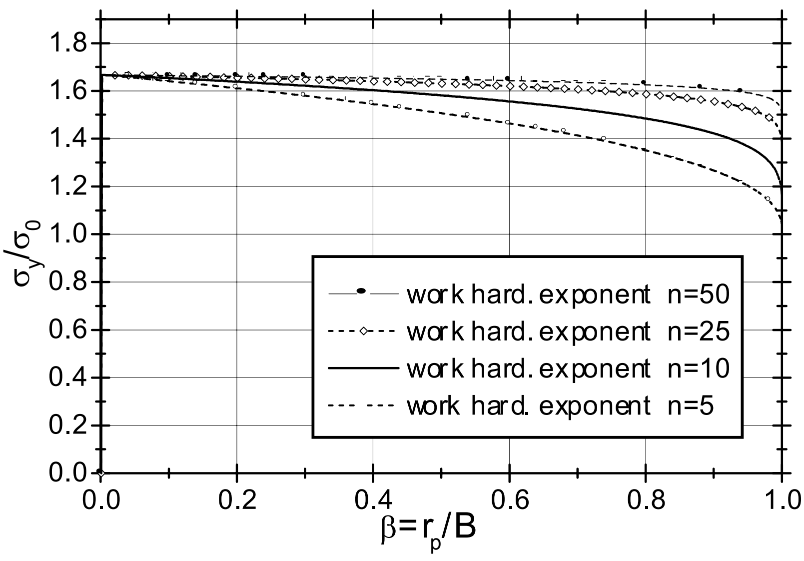


Figure 2. Yield stress distribution within the SYZ

where *e*=1.081, *f*=1.734, *c*=0.0968, *d*=0.976 in the case of the HMH hypothesis, *e*=0.999, *f*=1.564, *c*=0.102, *d*=0.983 in the case of the Tresca yield condition.

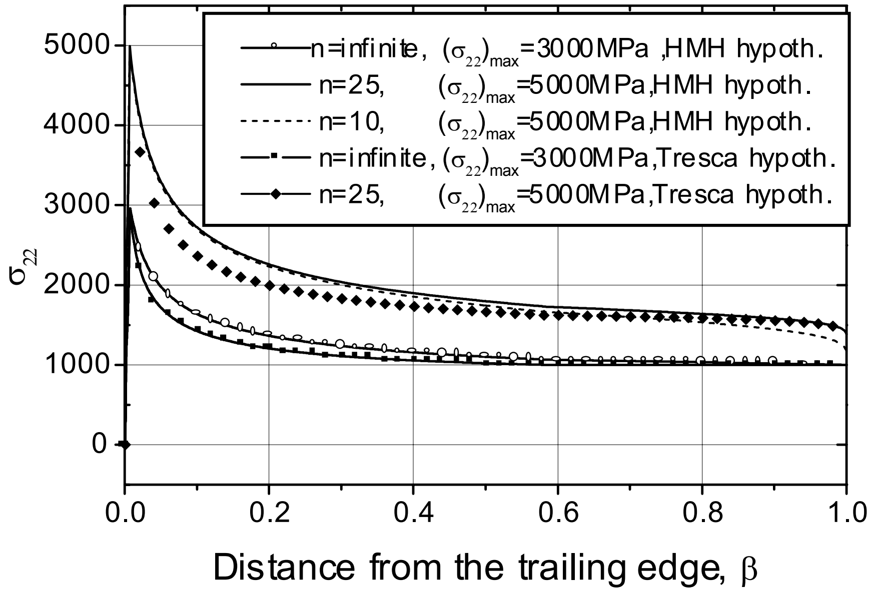


Figure 3. Stress distribution within SYZ according to HMH and Tresca hypotheses for $\gamma = r_p/B = 1$, $\sigma_0 = 1000$ MPa

CONCLUDING REMARKS

In the paper Dugdale model has been generalized to include *in* and *out-of-plane* constraints and work hardening properties of material. Constraints entered model through $\alpha(\beta=1)$ function – *in-plane* constraint and TASC factor *Tz* – *out-of-plane* constraint.

Computed stress distributions along SYZ (Fig.3) preserve the qualitative and to some extent quantitative features of the numerically computed [1-4] stress fields in front of the crack tip. Formula to determine the *average* stress distribution within SYZ, σ_{SYZ} can be expressed in a simple form (Eq. (13)). It preserves the simplicity of Dugdale model (σ_0 is simply replaced by σ_{SYZ}) for its practical applications within the framework of Fracture Mechanics. However, σ_{SYZ} contains "information" concerning geometrical constraints as well as work hardening properties of material.

To use the model one should know: yield stress σ_0, R-O power exponent *n*, specimen thickness *B*, ξ_n, ξ_∞ , which can be found in literature or computed numerically, *in-plane* constraint parameters: *T* or *Q* – stresses.

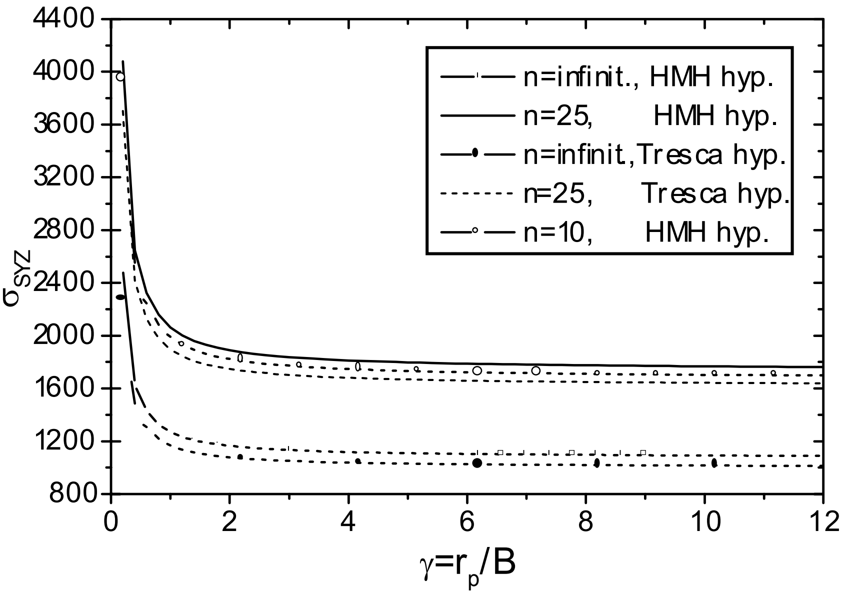


Figure 4. σ_{SYZ} as a function of relative length of the SYZ *for* $\xi_\infty=3$, $\xi_n=5$, $n=25$, $n=10$ *and* $n=\infty$ both for HMH and Tresca hypotheses.

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