Fracture Statistics of Brittle Materials: How to Choose a Better Distribution Function

Chunsheng Lu[1,2], Robert Danzer[1] and Franz Dieter Fischer[2]

[1] Institut für Struktur- und Funktionskeramik, Montanuniversität Leoben, A-8700, Austria
[2] Institut für Mechanik, Montanuniversität Leoben, A-8700, Austria

ABSTRACT: *A simple quantitative method in terms of the Akaike information criterion is proposed, which can be used to highlight the difference between the Weibull and normal or other favourite distributions, and further to find out which model is better. As an example, the fit of fracture data of brittle materials to several distributions, such as the Weibull, normal, and log-normal distributions, is compared. The results show that there seems to be no sufficient evidence that the Weibull distribution is always in preference to other distributions, and the uncritical use of the Weibull distribution is questioned. This is also verified by recent experiments on the fracture strength of electroceramics.*

INTRODUCTION

Measurement of the strength of brittle materials, such as ceramics, rock and concrete etc., typically produces considerable scatter in the results even if a set of nominally identical specimens are tested under the same conditions. Thus, both the description of strength and the assessment of reliability of brittle materials require a probability approach [1,2]. As is well known, the Weibull distribution has been found successfully in describing a large body of fracture strength data. As Weibull mentioned in his pioneering papers [3,4], however, the Weibull distribution should be considered as an empirical one on an equal footing with other distribution functions. As some possible candidates, the normal, log-normal, power law, and Gumbel distributions etc. could be involved [5].

In general, we attempt to identify an appropriate model for the data using the so-called goodness-of-fit tests. However, for a small sample size, it is difficult to distinguish between two distributions such as the Weibull and normal distributions. In this paper, we will propose a simple quantitative method, which can be used to highlight the difference between the Weibull and normal or other favourite distributions, and, furthermore, find out which model is better.

WEIBULL, NORMAL AND LOG-NORMAL DISTRIBUTIONS

The following three distributions are chosen and applied to the fit of strength data. Their probability density functions are [4,5] :

Weibull Distribution Function

$$p(\sigma) = \frac{m}{\sigma_0}\left(\frac{\sigma - \sigma_{th}}{\sigma_0}\right)^{m-1} \exp\left[-\left(\frac{\sigma - \sigma_{th}}{\sigma_0}\right)^{m}\right] \quad (1)$$

where σ_0 is a normalized strength, σ_{th} is the threshold stress (below which no failure will occur), and *m* is the Weibull modulus or shape factor.

Normal Distribution Function

$$p(\sigma) = \frac{1}{\sqrt{2\pi}\alpha} \exp\left[-\frac{(\sigma - \bar{\sigma})^2}{2\alpha^2}\right] \quad (2)$$

where $\bar{\sigma}$ and α are the mean and standard deviation, respectively.

Log-normal Distribution Function

$$p(\sigma) = \frac{1}{\sqrt{2\pi}\alpha\sigma} \exp\left[-\frac{(\ln\sigma - \bar{\sigma})^2}{2\alpha^2}\right] \quad (3)$$

where $\bar{\sigma}$ and α are the mean and standard deviation, respectively.

AKAIKE INFORMATION CRITERION

The best estimate of the unknown parameters in a distribution function is to apply the maximum likelihood method, which shows the smallest coefficient of variation whilst it is more cumbersome than the usually used linear-regression approach [2]. Here, the log-likelihood for a given probability density function is defined as $\ln L = \sum_{i=1}^{N} \ln p(\sigma_i)$, where σ_i is the strength of the ith sample, and N is the number of measurements. The solution is found by maximizing the log-likelihood, for example, in the case of the normal distribution in Eq. 2, so that $\partial \ln L / \partial \alpha = 0$ and $\partial \ln L / \partial \overline{\sigma} = 0$.

To compare strength data with distribution functions, some measure of the goodness-of-fit between the functions and data is required. The likelihood ratio statistics appears to be the most promising for use in obtaining confidence bounds. Following very similar considerations, the likelihood approach can be extended to making comparisons between distributions by the Akaike information criterion (AIC), which starts by linking the likelihood to a distance between the true and estimated distributions, and is defined as

$$AIC = -2(\ln \hat{L} - k) \quad (4)$$

where $\ln \hat{L}$ is the maximum log-likelihood for a given model, k is the number of parameters to be fitted in the model, and the additional factor 2 is a sop to historical precedents and could be omitted [6,7]. This represents a rough way of compensating for additional parameters and is a useful heuristic measure of the relative effectiveness of different models [8,9]. The best distribution is that for which AIC has the smallest value. In typical cases, differences between specific distributions, which would be significant at around the 5% level, correspond to differences in AIC values of around 1.5 ~ 2.

RESULTS AND DISCUSSION

As shown in Table 1, AIC_w , AIC_n, and AIC_{ln} are the AIC values calculated by the Weibull, normal, and log-normal distributions respectively, and the difference of AIC values is defined as $\Delta AIC = AIC_w - \min(AIC_n, AIC_{ln})$. Please note that the two-parameter Weibull distribution (i.e., let $\sigma_{th} = 0$ in Eq. 1) is used here. The influence of the threshold stress on the fit of the Weibull distribution has been discussed in [10]. It is obvious that, for the Si_3N_4 ceramics, the Weibull distribution fits the data better than the normal or log-normal distribution. For the SiC ceramics, both the Weibull and normal distributions fit the data better than the log-normal distribution. But the difference is not large enough to make a clear distinction between the Weibull and normal distributions. In the case of ZnO ceramics, the behavior

TABLE 1: AIC values calculated by the Weibull, normal, and log-normal distributions

Specimen	N	AIC_w	AIC_n	AIC_{ln}	ΔAIC
Si_3N_4	55	635.78	642.78	648.37	-7.00
SiC	75	778.31	779.68	785.60	-1.37
ZnO	109	681.29	671.53	672.52	9.76

is just opposite, and both the normal and log-normal distributions fit the data better than the Weibull distribution. But there is also not a clear distinction between the normal and lognormal distributions. Thus, the results show that the Weibull distribution is not always in preference to other distributions. On the other hand, based on as few assumptions as possible (i.e., weakest link hypothesis and no interaction between defects), a general strength distribution function has been suggested and represented as [11]

$$P_S(\sigma) = 1 - \exp\left[-\left\langle N_{c,S}(\sigma)\right\rangle\right] \tag{5}$$

where P is the cumulative distribution function, and $\left\langle N_{c,S}(\sigma)\right\rangle$ indicates the mean number of critical defects with length c in a specimen of size S. Obviously the Weibull distribution is only a special case of this general distribution function.

It is worth noting that statistical analysis is, after all, a kind of post-mortem method, and further mechanical or experimental evidence is needed. As is well known, size effect of strength is a direct consequence of the Weibull distribution: The larger the specimen, the higher the probability a large and critical defect could be found and the smaller its mean strength [12-14]. This can be represented as $V_1\sigma_1^m = V_2\sigma_2^m$ if we suppose two specimens with different sizes V_1 and V_2 as well as the same probability of failure. It provides us another way to check out the results mentioned above. As depicted in Figure 1, for the Si_3N_4 and SiC ceramics, the data follow the Weibull distribution, but in the case of ZnO ceramics, there is no size effect and the normal or log-normal distribution is a possible choice [15,16].

In fact, extensive investigations have shown that fracture of ceramics generally originates from defects. Microscopic observations indicated that there are very different kinds of flaws in these materials. In the Si_3N_4 and SiC ceramics, crack-like flaws are sparsely distributed, and thus it is not surprising that their strengths yield the Weibull distribution. But, in the ZnO

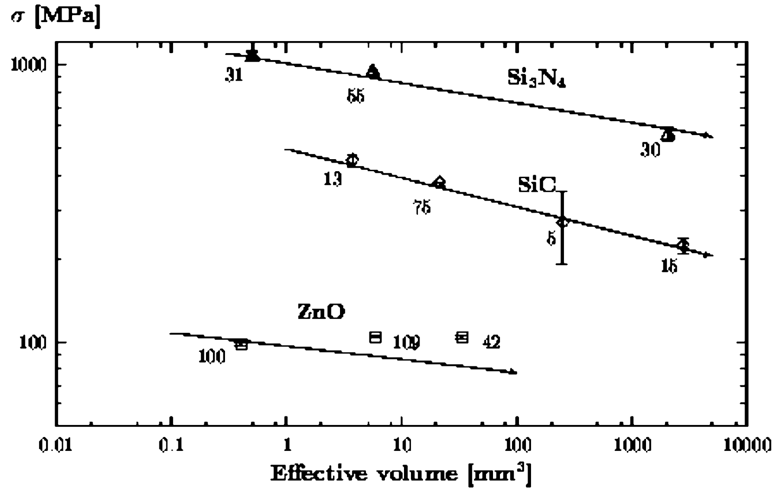

Figure 1: Experimental results for three ceramics, where numerals represent the number of specimens in the sample, error bars refer to 90% confidence band (the higher the number of tests, the smaller the scatter of data), and solid arrow lines, with the slope of $-1/m$, indicate the size effect extrapolated by the Weibull distribution.

ceramics, flaws are approximately spherical pores with sharp grooves. As we know, ZnO ceramics are a typical kind of electroceramics, which are used for varistors and designed with respect to electrical rather than mechanical properties. Thus, they contain a large number of flaws and have high porosity (about 5 vol%) which may act as the origin of fracture [15]. As a consequence, the final fracture may depend on many stochastic factors [17,18], and, according to the central limit theorem, the normal or log-normal distribution may be expected.

CONCLUSIONS

In conclusion, a simple quantitative procedure to ascertain a better distribution has been introduced, and further applied to the comparison of the fit of strength data of ceramics to the Weibull, normal, and log-normal distributions. The results show that there seems to be no sufficient evidence that the Weibull distribution is always in preference to the normal or other distributions. This is also verified by recent experiments on the fracture strength of electroceramics. The careful search for the better distribution could, however, provide the first clues and help us to elucidate the underlying physical mechanisms of fracture.

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