Thermal Stresses Around Interface Penny-Shaped Crack in a Laminated Composite

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ABSTRACT: *The present contribution deals with a two-layered composite structure weakened by an interface penny-shaped crack under the action of couple of concentrated heat sources at the vicinity of the defect. The corresponding 3D stationary thermoelasticity problem is formulated within the framework of linear thermoelasticity with microlocal parameters. By constructing the appropriate representations of the temperature, displacements and stresses through some harmonic functions, the resulting boundary value problem is reduced to some mixed problem of potential theory that leads to singular integro-differential equation of Newton's type. Its approximated solution is obtained by using the analogue of Dyson's theorem. The stress intensity factors are determined. The influence of the properties of the subsequent layers on SIF is examined.*

INTRODUCTION

The previous studies of interface crack problems for periodically laminated composites [1-3] showed that within a framework of homogenized model with microlocal parameters [5, 6] the solution does not exhibit non-realistic oscillatory singularities. In paper [2] the general method of solution of 3D crack problems of linear thermoelasticity with microlocal parameters was outlined. The present contribution deals with some specific thermal loading.

THE PROBLEM DESCRIPTION

Let us consider a microperiodically stratified space, in which repeated fundamental lamina of small height l consists of two homogeneous isotropic layers of height l_1 and l_2 with different thermomechanical properties. Let λ_j, μ_j, k_j, β_j / $(3\lambda_j+2\mu_j)$, denote Lame constants, thermal conductivity and coefficient of linear thermal expansion respectively of the layer of j-th kind.

The perfect bonding and ideal thermal contact between the subsequent layers is assumed with the exception of one interface where a penny-shaped crack of radius a exists.

The body is exerted by uniform tensile load p at infinity and two concentrated heat sources with intensity W, located symmetrically with respect to the plane of the crack (Fig.1). The crack is assumed to be free of traction and thermally insulated.

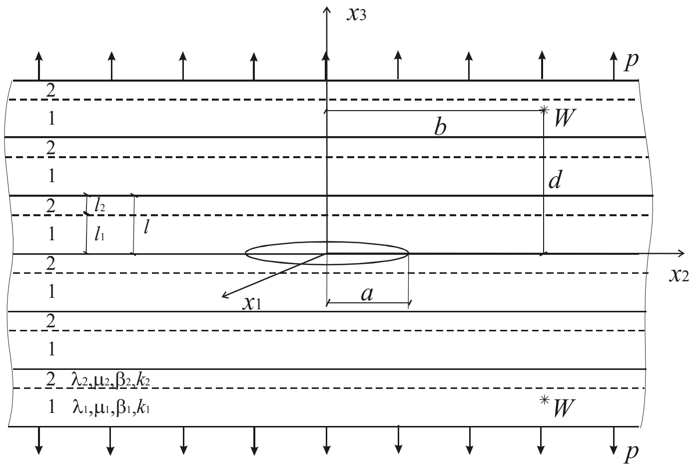

Figure 1: Two-layered periodically laminated medium with crack.

Let refer the body to the Cartesian coordinate system (x_1, x_2, x_3) with the centre being the centre of the crack and the x_3-axis normal to the layering. Let the coordinates of the points, where the heat sources act, be (*0*, *b*, *d*) and (*0*, *b*, *-d*).

The problem lies into determination of temperature, displacements and stresses distribution in the body. The stress field in a region close to the crack periphery is of special interest.

Formulation of the boundary value problem

For description of thermomechanical behaviour of the periodically stratified composite we will use the homogenized model with microlocal parameters [5, 6]. For the stationary case of linear thermoelasticity the governing equations of the homogenized model (after eliminating the microlocal parameters) take the form*

* The indices α, γ run over 1,2. Subscrips proceded by a comma indicate partial differentiation with respect the corresponding coordinates. Summation convention holds.

$$\tilde{k}\, t_{,\gamma\gamma} + K\, t_{,33} = -W\delta(x_1)\delta(x_2 - b)\delta(x_3 - d) - W\delta(x_1)\delta(x_2 - b)\delta(x_3 + d),$$
$$0,5\left(c_{11} + c_{12}\right) w_{\gamma,\gamma\alpha} + 0,5\left(c_{11} - c_{12}\right) w_{\alpha,\gamma\gamma} + c_{44} w_{\alpha,33} + \left(c_{13} + c_{44}\right) w_{3,3\,\alpha} = K_1\, t_{,\alpha},$$
$$\left(c_{13} + c_{44}\right) w_{\gamma,\gamma 3} + c_{44}\, w_{3,\gamma\gamma} + c_{33}\, w_{3,33} = K_3\, t_{,3}, \tag{1}$$

where t, $\boldsymbol{w}$ are the macrotemperature and the vector of macrodisplacement respectively. $\delta(\bullet)$ denotes the Dirac delta-function.

The stresses and heat fluxes in the body are expressed through the macrotemperature and the macrodisplacements vector in the form

$$\begin{aligned}
&\sigma_{\alpha 3} = c_{44}\left(w_{\alpha,3} + w_{3,\alpha}\right), \qquad \sigma_{33} = c_{13}\, w_{\gamma,\gamma} + c_{33}\, w_{3,3} - K_3\, t,\\
&\sigma_{12}^{(j)} = \mu_l\left(w_{1,2} + w_{2,1}\right),\\
&\sigma_{11}^{(j)} = d_{11}^{(l)}\, w_{1,1} + d_{12}^{(j)}\, w_{2,2} + d_{13}^{(j)}\, w_{3,3} - K_2^{(j)}\, t,\\
&\sigma_{22}^{(j)} = d_{12}^{(j)}\, w_{1,1} + d_{11}^{(j)}\, w_{2,2} + d_{13}^{(j)}\, w_{3,3} - K_2^{(j)}\, t,\\
&q_3 = -K\, t_{,3}, \qquad q_\alpha^{(j)} = -k_j\, t_{,\alpha}.
\end{aligned} \tag{2}$$

The constants appearing in the above equations depend on the material and geometrical characteristics of the subsequent layers and are given in the Appendix.

The considered penny-shaped crack problem is described by the following boundary conditions

$$\begin{aligned}
&\sigma_{33}(x_1, x_2, \pm 0) = 0, \qquad \sigma_{\alpha 3}(x_1, x_2, \pm 0) = 0,\\
&\qquad q_3(x_1, x_2, \pm 0) = 0, \qquad \forall (x_1, x_2) : \sqrt{x_1^2 + x_2^2} < a;\\
&\sigma_{33}(x_1, x_2, \pm\infty) = p, \qquad \sigma_{\alpha 3}(x_1, x_2, \pm\infty) = 0,\\
&\qquad q_3(x_1, x_2, \pm\infty) = 0, \qquad \forall (x_1, x_2) : \sqrt{x_1^2 + x_2^2} < \infty.
\end{aligned} \tag{3}$$

SOLUTION TO THE PROBLEM

For the boundary value problem stated in previous section the principle of superposition will be applied. So the general solution is separated into two parts. The first one corresponds to the uncracked space under action of described mechanical and thermal loading. For the second part the negative traction and heat flux generated from first part at the crack area are applied.

The uncracked periodically laminated medium with a couple of heat sources. In view of the fact that the uniform normal stresses at infinity induce normal traction in whole body and hence at the crack area, restrict our attention to consideration of thermal effects.

Let us examine the action of a couple of heat sources in the periodically laminated space. Due to axial symmetry of the problem apply the cylindrical coordinate system (r, φ, x_3) such that the points of heat sources belong to axis x_3 and locate symmetrically with respect to the coordinate system centre.

In this case the governing equations are [2]

$$
\begin{aligned}
&\tilde{k}(t_{,rr} + r^{-1}t_{,r}) + Kt_{,33} = -W\left(\delta(x_3 - d) + \delta(x_3 + d)\right)\delta(r)/2\pi r,\\
&c_{11}\left(w_{r,rr} + r^{-1}w_{r,r} - r^{-2}w_r\right) + c_{44}w_{r,33} + \left(c_{13} + c_{44}\right)w_{3,3r} = K_1\, t_{,r},\\
&\left(c_{13} + c_{44}\right)\left(w_{r,r3} + r^{-1}w_{r,3}\right) + c_{33}\, w_{3,33} + c_{44}\left(w_{3,rr} + r^{-1}w_{3,r}\right) = K_3\, t_{,3}\,.
\end{aligned}
\tag{4}
$$

The solution can be found by using the method of integral transforms. Let represent the quantities wanted in form

$$
\begin{aligned}
t(r,x_3) &= \int_0^\infty \rho J_0(\rho r)d\rho \int_0^\infty \theta(\rho,\zeta)\cos(\zeta(x_3 - d))d\zeta,\\
w_r(r,x_3) &= \int_0^\infty \rho J_1(\rho r)d\rho \int_0^\infty \omega_r(\rho,\zeta)\cos(\zeta(x_3 - d))d\zeta,\\
w_3(r,x_3) &= \int_0^\infty \rho J_0(\rho r)d\rho \int_0^\infty \omega_3(\rho,\zeta)\sin(\zeta(x_3 - d))d\zeta.
\end{aligned}
\tag{5}
$$

Substituting the expressions Eq. (5) into governing equations Eq. (4) one arrives at system of linear algebraic equations with respect to the functions $\theta(\rho,\zeta)$, $\omega_r(\rho,\zeta)$, $\omega_3(\rho,\zeta)$, from which it is easy to obtain

$$
\begin{aligned}
\theta(\rho,\zeta) &= W/2\pi^2\tilde{k}\ (\rho^2 + t_0^{-2}\zeta^2)^{-1},\\
\omega_r(\rho,\zeta) &= \frac{W}{2\pi^2\tilde{k}\, c_{11}c_{44}}\frac{\rho\left[K_1c_{44}\rho^2 + (K_1c_{33} - K_3c_{13} - K_3c_{44})\zeta^2\right]}{(\rho^2 + t_0^{-2}\zeta^2)(\rho^2 + t_1^{-2}\zeta^2)(\rho^2 + t_2^{-2}\zeta^2)},\\
\omega_3(\rho,\zeta) &= \frac{W}{2\pi^2\tilde{k}\, c_{11}c_{44}}\frac{\zeta\left[(K_3c_{11} - K_1c_{13} - K_1c_{44})\rho^2 + K_3c_{44}\zeta^2\right]}{(\rho^2 + t_0^{-2}\zeta^2)(\rho^2 + t_1^{-2}\zeta^2)(\rho^2 + t_2^{-2}\zeta^2)},
\end{aligned}
\tag{6}
$$

where $t_0 = \sqrt{\tilde{k}/K}$ and t_1, t_2 are positive roots of the biquadratic equation

$$c_{44}c_{33}t^4 - (c_{11}c_{33} - c_{13}^2 - 2c_{13}c_{44})t^2 + c_{11}c_{44} = 0\,. \tag{7}$$

Performing straightforward calculations and making use of Eqs. (6), (5), (2), the macrotemperature and macrodisplacement vector and hence the components of stress tensor and heat flux vector can be easily obtained:

$$t(r,x_3) = \frac{W}{4\pi K}\left(R_{i-}^{-1} + R_{i+}^{-1}\right),\ q_3 = \frac{W\,t_0^2}{4\pi}\left((x_3-d)R_{i-}^{-3} + (x_3+d)R_{i+}^{-3}\right),$$

$$\sigma_{r3} = \frac{-W}{4\pi K c_{33}}\sum_{i=0,2}\frac{\left[(K_1c_{13} - K_3c_{11}) + (K_1c_{33} - K_3c_{13})t_i^2\right]}{\prod\limits_{\substack{j=0,2\\ j\neq i}}(t_j^2 - t_i^2)}\frac{t_i}{r}\left(\frac{(x_3-d)}{R_{i-}} + \frac{(x_3+d)}{R_{i+}}\right),$$

$$\sigma_{33} = \frac{W}{4\pi K c_{33}}\sum_{i=0,2}\frac{\left[(K_1c_{13} - K_3c_{11}) + (K_1c_{33} - K_3c_{13})t_i^2\right]}{t_i\prod\limits_{\substack{j=0,2\\ j\neq i}}(t_j^2 - t_i^2)}\left(\frac{1}{R_{i-}} + \frac{1}{R_{i+}}\right), \tag{8}$$

where $R_{i\pm}$ denotes $R_{i\pm} = \sqrt{r^2 + t_i^2(x_3 \pm d)^2}$.

The remaining components of heat flux vector and stress tensor are not of immediate interest for further consideration.

The crack problem.

In order to obtain the second part of the solution it is necessary to apply the negative traction and heat flux generated by first part in the area occupied by the crack. Taking into account Eqs.(8) (in which it must be put $r = \sqrt{x_1^2 + (x_2 - b)^2}$) it easy to see that conditions $(3)_2$ and $(3)_3$ are automatically satisfied. It means that in case under consideration the crack does not disturb the temperature field, but the mechanical field.

For determination of disturbed mechanical field we will use the representation of stresses and displacements within a two-component periodically laminated composite through harmonic functions, supposed in [2]. Their form depends on the properties of subsequent layers. For case $\mu_1 \neq \mu_2$ the normal displacements and normal stresses at the plane $z = 0$ have the form

$$\begin{aligned} w_3(x_1,x_2,0) &= \left[m_2(1+m_2)^{-1} - m_1(1+m_1)^{-1}\right] f_{,3}(x_1,x_2,0),\\ \sigma_{33}(x_1,x_2,0) &= c_{44}\left(t_2^{-1} - t_1^{-1}\right) f_{,33}(x_1,x_2,0); \end{aligned} \tag{9}$$

where $f(\boldsymbol{x})$ is a harmonic function and $m_\alpha = \left(c_{11}t_\alpha^{-1} - c_{44}\right)\big/\left(c_{13} + c_{44}\right)$.

Representing the unknown harmonic function $f(\mathbf{x})$ as a potential of single layer with density $\phi(x_1, x_2)$ and satisfying the boundary condition $(3)_1$, one arrives at the singular integro-differential equation

$$\Delta \iint\limits_S \frac{\phi(\xi_1, \xi_2)\, d\xi_1\, d\xi_2}{\sqrt{(x_1-\xi_1)^2 + (x_2-\xi_2)^2}} = -2\pi p - \sum_{i=0}^{2} \frac{F_i}{\sqrt{x_1^2 + (x_2-b)^2 + t_i^2 d^2}} \quad ,(10)$$

where $\Delta(\bullet) = (\bullet)_{,11} + (\bullet)_{,22}$, $F_i = \dfrac{W\left[(K_1 c_{13} - K_3 c_{11}) + (K_1 c_{33} - K_3 c_{13}) t_i^2\right]}{K c_{33}\; t_i \prod\limits_{j=0,2;\, j\neq i} (t_j^2 - t_i^2)}$.

For solution of Eq. (10) we will use the analogue of Dyson's theorem [4]. To this end, let expand the RHS of Eq. (10) in series of polynomials and then the function $\phi(x_1, x_2)$ present in the form

$$\phi(x_1, x_2) = \sqrt{a^2 - x_1^2 - x_2^2} \left(\psi + \sum_{i=0}^{2} \varphi_i \right) \tag{11}$$

where ψ, φ_i denote polynomials with unknown coefficients. In accordance with the statement of Dyson's theorem the integral in the left side of Eq. (10) is a polynomial. From equality of two polynomials the system of linear algebraic equations for unknown coefficients of ψ, φ_i is obtained.

The calculations were performed for polynomials of 4th degree. Thus,

$$\psi = \frac{2p}{\pi}; \qquad \begin{aligned} \varphi_i = {} & a_{00}^{(i)} + a_{10}^{(i)}(x_1^2 + x_2^2) + a_{01}^{(i)} x_2 + \\ & + a_{11}^{(i)}(x_1^2 + x_2^2) x_2 + a_{20}^{(i)}(x_1^2 + x_2^2)^2 + a_{02}^{(i)} x_2^2 + a_{03}^{(i)} x_1^2, \end{aligned} \tag{12}$$

where

$$a_{00}^{(i)} = \frac{F_i}{\pi^2}\left(-1 + \frac{a^2}{18(b^2 + t_i^2 d^2)} - \frac{25a^2 b^2 + 4a^4}{150(b^2 + t_i^2 d^2)^2} \right);$$

$$a_{10}^{(i)} = \frac{F_i}{\pi^2}\left(\frac{2}{9(b^2 + t_i^2 d^2)} - \frac{4a^2}{75(b^2 + t_i^2 d^2)^2} \right);$$

$$a_{01}^{(i)} = \frac{F_i}{\pi^2}\left(\frac{-2b}{3(b^2 + t_i^2 d^2)} + \frac{4a^2 b}{15(b^2 + t_i^2 d^2)^2} \right);$$

$$a_{20}^{(i)} = -\frac{F_i}{\pi^2} \frac{8}{75(b^2 + t_i^2 d^2)^2}; \quad a_{11}^{(i)} = \frac{F_i}{\pi^2} \frac{8b}{15(b^2 + t_i^2 d^2)^2};$$

$$a_{02}^{(i)} = -\frac{F_i}{\pi^2}\frac{11b^2}{15(b^2+t_i^2d^2)^2}; \; a_{03}^{(i)} = \frac{F_i}{\pi^2}\frac{b^2}{15(b^2+t_i^2d^2)^2}.$$

So the complete stress-displacement field can be found from the harmonic function $f(\mathbf{x})$ by using (11), (12).

The crack-border stress field features an inverse square root singularity and is characterized by the stress intensity factor (SIF).

The SIF K_I, defined in the conventional manner $K_I = \lim_{s\to+a}\sqrt{2(s-a)}\sigma_{33}$, can be expressed through the functions ψ, φ_i in the form [4] $K_I = 2\pi\sqrt{\pi a}\left(\psi + \sum_{i=0}^{2}\varphi_i\right)$.

In order to examine the dependence of the stress intensity factor on the geometrical and physical properties of the subsequent layers, two simplifying assumptions were made: $\mu_i = \lambda_i$ and $k_1 = k_2$. The following dimensionless parameters were introduced: $\eta = l_1/l$, $\gamma = \mu_2/\mu_1$, $\alpha = \beta_2/\beta_1$, $K_I{*} = K_I/\sqrt{a}p$, $W^* = W\beta_1/k_1p$. The dependence of SIF K_I on parameters γ and α are shown in Fig.2 and Fig.3, respectively. The calculations were carried out for the case $b/a = 0$.

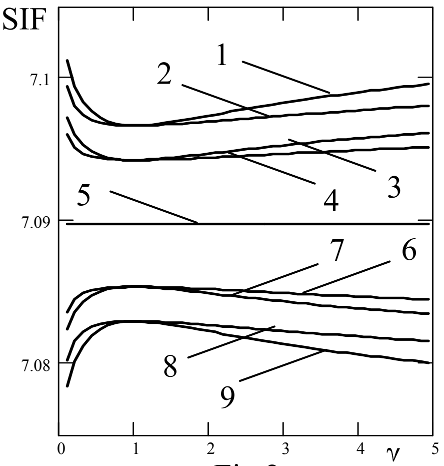

Fig.2.

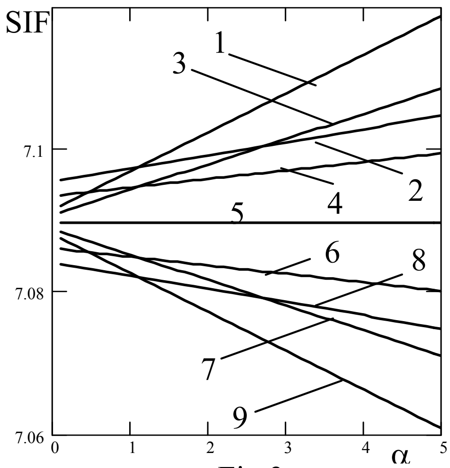

Fig.3.

1 – η=0.5, *d/a*=4, W*=-0.1; 2 – η=0.1, *d/a*=4, W*=-0.1
3 –η =0.5, *d/a*=5, W*=-0.1; 4 – η=0.1, *d/a*=5, W*=-0.1
5 – W*=0
6 – η=0.1, *d/a*=5, W*=0.1; 7 – η=0.5, *d/a*=5, W*=0.1
8 – η=0.1, *d/a*=4, W*=0.1; 9 – η=0.5, *d/a*=4, W*=0.1

APPENDIX

Denoting by $b_j = \lambda_j + 2\mu_j \ \left(j = 1,2\right)$, $b = \left(1-\eta\right) b_1 + \eta\, b_2$, the positive coefficients in equations (1), (2) are given by the following formulae:

$$c_{33} = \frac{b_1\, b_2}{b}, \quad c_{11} = c_{33} + \frac{4\eta\left(1-\eta\right)\left(\mu_1 - \mu_2\right)\left(\lambda_1 - \lambda_2 + \mu_1 - \mu_2\right)}{b},$$

$$c_{13} = \frac{\left(1-\eta\right)\lambda_2\, b_1 + \eta\,\lambda_1\, b_2}{b}, \quad c_{44} = \frac{\mu_1\mu_2}{\left(1-\eta\right)\mu_1 + \eta\,\mu_2},$$

$$c_{12} = \frac{\lambda_1\lambda_2 + 2\left[\eta\,\mu_2 + \left(1-\eta\right)\mu_1\right]\left[\eta\,\lambda_1 + \left(1-\eta\right)\lambda_2\right]}{b},$$

$$d_{11}^{(l)} = \frac{4\mu_l\left(\lambda_l + \mu_l\right) + \lambda_l\, c_{13}}{b_l}, \quad d_{12}^{(l)} = \frac{2\mu_l\lambda_l + \lambda_l\, c_{13}}{b_l}, \quad d_{13}^{(l)} = \frac{\lambda_l\, c_{33}}{b_l},$$

$$K_1 = \frac{\eta\beta_1\lambda_2 + (1-\eta)\beta_2\lambda_1 + 2\left[\eta\,\mu_2 + \left(1-\eta\right)\mu_1\right]\left[\eta\,\beta_1 + \left(1-\eta\right)\beta_2\right]}{b},$$

$$K_3 = \frac{\eta\beta_1 b_2 + (1-\eta)\beta_2 b_1}{b}, \quad K = \frac{k_1 k_2}{(1-\eta)k_1 + \eta k_2},$$

$$K_2^{(j)} = \frac{2\beta_j\mu_j + \lambda_j K_3}{b_j}, \quad \tilde{k} = (1-\eta)k_1 + \eta k_2 .$$

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