The SIF Calculation of Internal Cracks under Rolling Contact Loads by a Weight Function Approach

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ABSTRACT: *The problem of the SIF determination of internal cracks under contact loading is dealt with. An approach based on the weight function method was developed for the determination of K_I and K_{II} . The effect of the contact and of friction between the crack faces was taken into consideration by using approximate relations. The results were compared with the ones obtained by finite element analyses and the agreement was found satisfactory, both without and with the contact between the crack faces.*
As possible application of the approach, the case of contact fatigue crack propagation in a automotive gear was dealt with.

INTRODUCTION

Contact analysis is becoming more and more important in the design of many mechanical systems like gears and bearing.

One of the most important criteria for dimensioning these elements is the contact fatigue damage, which consists in the removal of surface material due to the stable advance of a microcrack till the surface of the element. It is caused by the cyclic repetition of the contact stresses and its origin can be either on the surface or under it, generally in correspondence of a hard inclusion.

If the attention is focussed on the sub-surface cracks all the references recognise that the crack propagation is governed by the cyclic variation of the stress intensity factor concerning the mode *II* (sliding). This calculation is, nevertheless, complex. In fact, due to the nature of the applied load, the crack faces are in contact and no analytical solution is available taking into account the friction between the crack faces. Kaneta et al. [1] proposed a three-dimensional approach enabling the calculation of the SIF for sub-surface circular cracks, but a limited number of cases were published.

More recently some approaches [2, 3] based on Finite Element method able to calculate the stress intensity factors and to predict the direction and the rate of the crack growth were presented. However, all these FE approaches are expensive and time-consuming.

As a consequence of this, particular efforts have been recently dedicated to the research of alternative, no time-consuming numerical methods, able to give accurate solutions of practical cases in a reasonable time. Among these ones the Weight Function (WF) method is becoming more and more popular because of its versatility and low time needed for its application

In this paper it is described an approach to determine the weight functions of an internal crack under mode *I* and *II* loading. The WF is based on the results obtained in [4], relating a symmetric load condition, but it is applied to non-symmetric load cases. The friction between the crack faces was also considered in an approximate way: the comparison with FE results allowed judging the accuracy of the results. One example of an application to contact fatigue design practice is provided, namely the analysis of crack propagation in treated gears.

THE WEIGHT FUNCTION METHOD

The weight function method was introduced by Bueckner [5, 6], that showed how to determine the SIF of a cracked body by integrating over the crack length, *a*, the product of the Weight Function, *m(a,x)*, for the stress pattern of the non cracked body, $\sigma(x)$:

$$K = \int_0^a m(a,x)\sigma(x)dx \qquad (1)$$

The major advantage of the weight function method is that the weight function *m(a,x)* is a function of the geometry of the cracked body; so, once it is known the SIF of a cracked body can be determined by knowing the stress distribution of the uncracked body over the crack length.

For the calculation of the WF, it can be demonstrated the validity of the following formula:

$$m(a,x) = \frac{E'}{K_I^r(a)} \frac{\partial v^r(a,x)}{\partial a} \qquad (2)$$

where: $E'=E$ for plane stress state; $E' = \frac{E}{(1-\nu^2)}$ for plane strain state; $v^r(x,a)$ and K_I^r are respectively the COD and the mode *I* SIF due to a reference load system as a function of the crack length *a*.

In order to determine the Weight Function it is, therefore, necessary to know the *K(a)* and *v(a,x)* at least for a single reference case [7, 8].

In particular for the case considered in this paper (internal crack in a half-plane, parallel to the external surface and subjected to a rolling contact loading) it is possible to define two reference loading systems. The first for the Weight Function relating to the mode *I* is a uniform pressure, p_o, on the crack faces, as it is shown in Figure 1a, the second for the Weight Function relating to the mode *II* is a tangential stress uniform, τ_o, on the crack faces, as it is shown in Figure 1b.

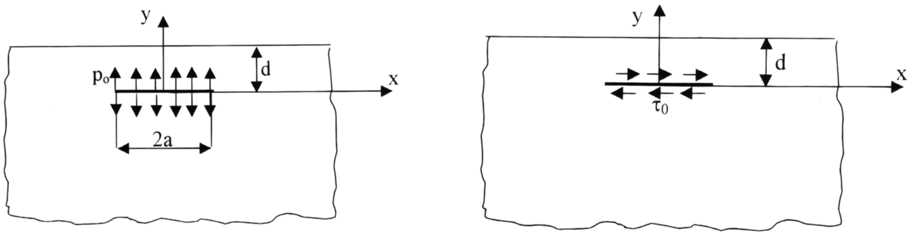

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Figure 1: Reference loading system: a) Weight Function related to the mode *I*; b) Weight Function related to the mode *II*.

The problem is, therefore, the determination of the SIF of a crack internal in an elastic half-plane parallel to the external surface and loaded by a uniform pressure and by a uniform shear stress.

The case was solved by referring to the numerical analysis proposed by Erdogan and Arin [4], that is based on the solution of a system of integral equations of Fredholm. This solution is related to a general case of an inclined crack in a domain and has been reduced to the case of a crack parallel to the external surface.

The equations obtained by the application of the method of Erdogan are reduced to an algebraic equation system, by extending in series by Chebyshev polynomial both the equation terms. Further details will be found in [9].

In Figures 2a and 2b the patterns of the reference non-dimensional $K^r{}_I$ and $K^r{}_{II}$ for the cases of Figure 1 are reported. Indeed if p_o is applied also a K_{II} is present, but in the case analysed the values are negligible and are not reported. It is possible to note that if $d/a \to 0$ the $K \to \infty$ and therefore the values meaning less. Therefore, it is important to define some validity limits, in particular if $d/a \leq 0.4$ the corresponding values of SIFs are not considered reliable.

In order to determine the Weight Function it is, now, necessary to calculate the reference displacement functions. In [10] the displacement components u^r and v^r, respectively in direction x and y, are reported. In particular, for the case of a crack loaded by a uniform pressure p_o the expressions are:

$$u^r_{(p_o)}(x) = -\frac{1-\nu}{E} p_0 x \qquad v^r_{(p_o)}(a,x) = \frac{2 p_0 a}{E}\left(1-\frac{x^2}{a^2}\right)^{\frac{1}{2}} \tag{4}$$

and for the case of a crack loaded by uniform tangential stress τ_o are:

$$u^r_{(\tau_o)}(a,x) = \frac{2\tau_0 a}{E}\left(1-\frac{x^2}{a^2}\right)^{\frac{1}{2}} \qquad v^r_{(\tau_o)}(x) = -\frac{1-\nu}{E}\tau_0 x \tag{5}$$

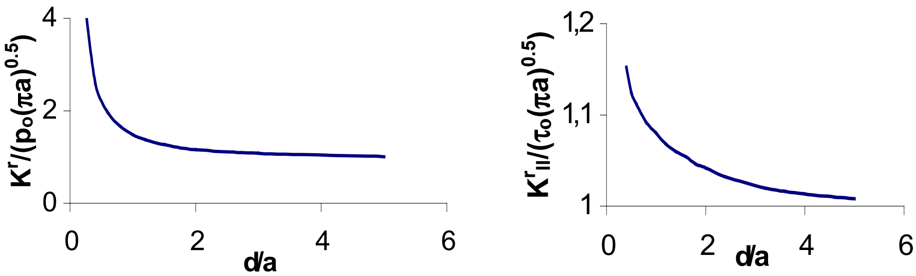

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Figure 2: Non dimensional reference SIFs for the cases of internal crack: a) under uniform pressure p_o; b) under uniform tangential stress τ_o.

It is possible, now, to evaluate the Weight Function for the case related to Figure 1a:

$$m_1 = \frac{E\left(\frac{\partial}{\partial a} v^r_{(p_o)}(a,x)\right)}{K^r_I(d/a)} \tag{6}$$

Where $v^r_{(p_o)}$ and K^r_1 are defined respectively in Eq.4 and in Figure 2.

By replacing and deriving it follows:

$$m_1(a,x) = \frac{2}{K^r_I(d/a)\sqrt{\pi a}\sqrt{\frac{a^2-x^2}{a^2}}} \tag{7}$$

In order to obtain the Weight Function related to case of Figure 1b the same proceeding is followed:

$$m_2(a,x) = \frac{E\left(\frac{\partial}{\partial a} u^r_{(\tau_o)}(a,x)\right)}{K^r_{II}(d/a)} \tag{8}$$

Where $u^r_{(\tau_o)}(a,x)$ and $K^r_{II}(a,x)$ are defined respectively in Eq.5 and Figure 2. By substituting and deriving it follows:

$$m_2(a,x) = \frac{2}{K^r_{II}(d/a)\sqrt{\pi a}\sqrt{\frac{a^2 - x^2}{a^2}}} \tag{9}$$

CALCULATION OF SIF UNDER ROLLING CONTACT LOADS

Once the WF is known, it is possible to calculate the SIF for an internal crack under rolling contact loads, by using (1). In particular in this paper the SIF concerning the mode *II* is dealt with. The stress distribution considered for the calculation of K_{II} is the one calculated by Smith and Liu [11] in uncracked elastic half-space, loaded by hertzian pressure distribution.

To determine the K_{II} values during an entire load cycle, the analysis was executed by varying the distance between the centre of the pressure distribution and the centre of the crack (distance *e* in Figure 3). In the case of $e \neq 0$, however, the WF found from the above load case is no more valid, since the problem lost the symmetry and mode *II* and *I* are now coupled.

However it was hypotized that for cracks of small dimensions and little values of *e* the solution found can be applied with an acceptable approximation from a practical point of view. The problem is that by using the solution it is not possible to distinguish between the crack tips, since the result is the same. To overcome this difficulty the values of the stress intensity factors for the two crack tips were calculated from the one obtained from the WF by weighting the stress distribution on the crack length. In other words, if K_{II} is the value obtained by the integration of the Weight Function, x_G is distance of the stress distribution barycentre from the centreline of the crack, it is defined a factor *p*:

$$p = \frac{a + x_G}{2a} \tag{10}$$

the SIF values in the tips A and B are, respectively:

$$K_{IIA} = pK_{II} \qquad K_{IIB} = (1-p)K_{II} \tag{11}$$

The values obtained by using this approach were verified with the results of finite element analyses, details are included in [9]. The crack length considered is $a=0.07mm$, the crack distance is $d=0.14mm$ and the value of the maximum pressure is $p_o= 1400\ MPa$. The complete rolling was simulated and the results from the two different approaches are shown in Figure 4.

The difference of the results by the two methods is lower than the 10%.

The comparisons were conducted even for different crack lengths and the results are shown in Table 1 in terms of variation of SIF in a complete loading cycle, ΔK_{II}, as it is defined in Figure 4.

It is evident that the error increases with the crack dimension. This is probably due to the hypothesis considered to determine the m_2 that are less and less respected if the crack length grows.

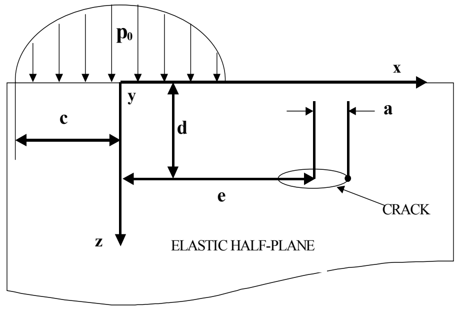

Figure 3: Scheme of the internal crack under rolling contact loading.

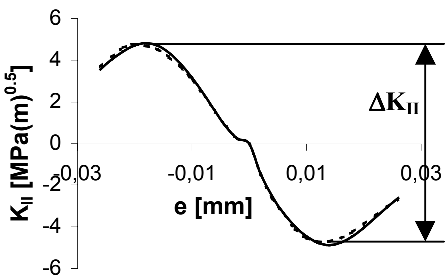

Figure 4: Simulation of a complete rolling: *a=0.07mm*, *d=0.14mm, f=0,* p_o*=1400 MPa*;------FE; ——— WF.

Besides m_2 was evaluated without considering the combined effect of both the crack propagation modes, that when the cracks become larger can be important.

TABLE 1Comparison of the ΔK_{II} results obtained by WF and FE analysis, *d=0.14mm*, p_o*=1400 MPa*, *f=0,0.4*

Crack length a [mm]	ΔK_{II} [$MPa\sqrt{m}$] WF		ΔK_{II} [$MPa\sqrt{m}$] FE	
	f=0	f=0.4	f=0	f=0.4
0.03	5.9	3.3	6.0	3.5
0.04	6.5	-	6.5	-
0.05	7.2	3.8	7.2	3.9
0.07	9.7	4.6	9.5	4.5
0.09	9.1	4.9	9.5	4.8
0.12	10.5	-	10.7	-
0.15	11.6	-	11.1	-
0.18	12.5	-	11.9	-
0.22	11.4	-	12.1	-
0.27	5.5	-	13.1	-
0.35	8.1	-	13.8	.

EFFECT OF THE FRICTION BETWEEN THE CRACK FACES

In view of a possible application to cases of practical interest, it is necessary to consider in the calculation the effect of the contact between the crack faces and of the consequent friction forces.

In this paper an approximate approach, based on the one described in [12], is applied.This latter assumes the same m_2 function but a modified stress distribution, τ_{eff}, used to calculate K_{II}.

In the Table 1 the comparison between the values of the variation of SIF in a complete loading cycle, ΔK_{II}, calculated by means of the Weight Function and the Finite Element model, by considering the friction effect, is shown.

The comparison is satisfactory and it is possible to consider available the method adopted to consider the effect of the friction coefficient.

APPLICATION TO CONTACT FATIGUE CRACK PROPAGATION

As application of the present approach a case of practical interest is considered: the contact fatigue crack propagation of an internal crack in an automotive spur gear tooth (module *3 mm*, maximum Hertz contact pressure $p_o=1400\ MPa$, material: carburized steel). The problem is nowadays of great interest, because recent researches show that the most of the life span of contact elements is spent during the crack propagation phase [13].

It is possible to find different laws in literature: the one considered in this paper is the law proposed by Glodez et al. [13].

The initial dimension of the crack, $2a=0.05\ mm$, that corresponds to the typical grain size, was assumed. The depth of the crack considered is $d=0.14\ mm$, which corresponds to the depth at which there is the maximum hertzian shear stress.

The value of ΔK_{II} was calculated by using the WF approach, taking into consideration the contact between the crack faces and assuming $f=0.4$. On the basis of the results of the analyses the following interpolation function was obtained:

$$\Delta K_{II} = 153a^3 - 123.25a^2 + 33.96a + 2.55 \quad [MPa\sqrt{m}] \qquad (15)$$

The calculation was stopped for $a=0.35mm$. In fact, for larger cracks it was shown that the numerical results are not reliable.

In Figure 5 it is shown the crack grow rate: the strong discontinuities are due to the deceleration of the crack near a grain boundary. The crack length trend versus the number of cycles is shown in Figure 6.

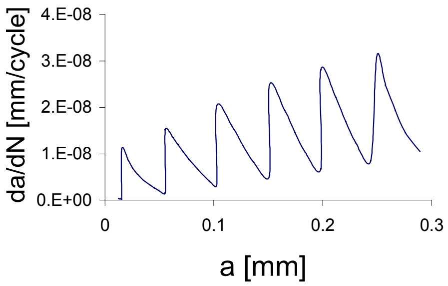

Fig.5: Crack growth rate.

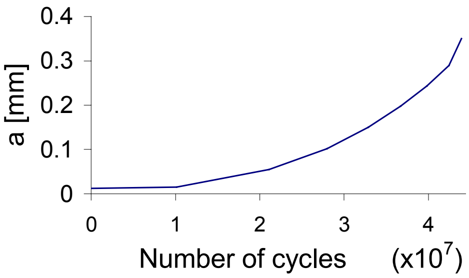

Fig.6: Crack length trend versus number of cycles.

CONCLUSIONS

A WF approach to calculate the SIF of an internal crack under rolling contact loading was presented. It is based on the solution found by Erdogan and allows taking into account the friction of the crack faces. It was applied to determine the SIF variation during an entire load cycle: since the load case is not symmetric when the pressure distribution is eccentric with respect to the crack tip, the range of validity of the approach was discussed.

On the basis of the results it is possible to affirm that the approach is accurate when the crack dimensions are limited. Since this is the dimensional range in which mechanical components spend most of their lifetime, the approach can be a useful tool to predict their duration.

REFERENCES

1. Kaneta M. et al. (1986) *Trans. of ASME, Journal of Tribology*, **108** pag. 134-139.
2. Bastias P.C. et al.(1989) *Engineering Fracture Mechanics*, **33**, n°1, pag. 143-152.
3. Komvopoulos K., Cho S. S., (1997) *Wear*, **209**, pag. 57-68, 1997.
4. Erdogan F., Arin K. (1975) *Int. Journal of Fracture*, **11**, n°2 , pag. 191-204.
5. Bueckner H. F., (1970) Zeitchrift fur angewandte Mathematik und Mechanik, **50**, pag. 529-546.
6. Bueckner H. F., (1971) Zeitchrift fur angewandte Mathematik und Mechanik, **51**, pag. 97-109.
7. Petroski, H.J., Achenbach, J.D. (1978) *Engineering Fracture Mechanics*, **10**, 257-266.
8. Beghini, M. et al. (1991) *Int. J. for Numerical Methods in Eng.,* **32**, pp.595-607.
9. Guagliano, M., Vergani, L. (2001) Internal Report CDR N. 47-2000, Dept. Mechanical, Politecnico di Milano.
10. Chow C.L., Lau K.J. (1976) *Int. J. of Fracture*, **12**, N°5, p.669-684
11. Smith J.O., Liu C. K., (1953) *ASME J. of Applied Mechanics*, **20**, pag.157-166.
12. Fletcher D. I., Beynon J. H., (1999) Proc. Instn. Mech. Engrs., **213**, pag. 481-486.
13. Glodez S. et al. (1997) *Fat. Fract. Engng. Mater. Struct.,* **20**, n°1, pag. 71-83.