NEW FRACTURE CRITERIA FOR ANISOTROPIC BODIES WITH CRACKS

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ABSTRACT: *New approach to strength analysis of thin-walled structural elements of anisotropic materials weakened by cracks under conditions of plane stress and bending is stated. The fracture criterion is grounded on the notion of the material damage measure in the process zone at the crack tip and on the assumption of the crack growth towards the highest possible value of this measure and its reaching the critical value. The criterion is employed in the investigation of the critical state of the highly anisotropic plate with a crack. With the change of the loading angle in the plates with high anisotropy the instability of initial crack motion angle is possible. The analysis of the examples demonstrates the quantitative and qualitative impact of the proposed generalizations on the initial cracking angle and the strength of the structure.*

INTRODUCTION

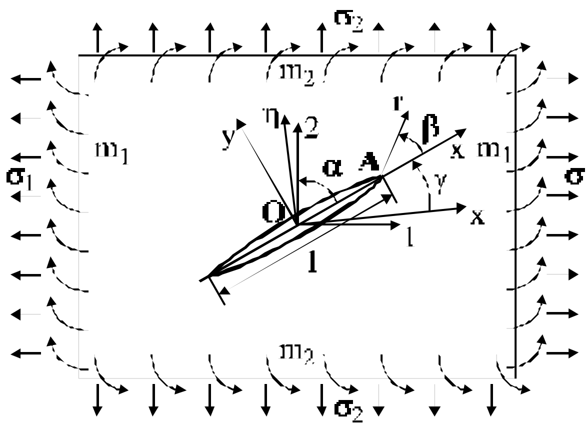

Figure 1: Scheme of the problem.

Plates of anisotropic materials, in particular, of reinforced concrete or composites, are popular elements of building constructions. They are widely used in machine building, transport construction etc. The main cause of their failure is the growth of cracks, either already existing in the manufactured product or arising in it and reaching the critical size on account of fatigue. The strength and reliability analysis in all cases envisages the use of materials brittle fracture methods. These methods are well known and approved by scientific and engineering practice with respect to isotropic materials [1]. There is a deficiency of experimental data concerning the parameters of static crack growth resistance [2]. Far less attention has been paid to working out fracture criteria of anisotropic materials with cracks [2, 3]. The practice of formulating the fracture criteria of anisotropic materials with cracks amounted to nothing more than the use of isotropic materials criteria, in which the account of the influence of anisotropy was reduced to the formal orientation dependence $K_{Ic} = K_{Ic}\left(b+g\right)$.

One or several methods of analysis belonging to one of the three main criteria classes (energy, force and strain ones) are used depending on the character of loading, type of material, availability of experimental data concerning mechanical and strength characteristics of the used material, peculiarities of the structure and its function. Force s_{qq}-criterion or criterion of maximal intensity of singular part of tensile stresses (F.E. Erdogan, G.S. Sih, G.P. Cherepanov, V.V. Panasyuk, L.T. Brezhnytskiy), being generalization of Irwin's criterion, is one of the simplest but universal enough, effective and experimentally approved criteria.

This paper presents fracture criteria of anisotropic plates with cracks, which are grounded on a new conception for notion of material damage in the process zone. Normalization of the stress levels, strains or energy by means of resistance ability provides more a reasonable evaluation on the material resistance as far as it does not depend on the stress-strained state but on its relationship with crack growth resistance. Such measures may be considered as monotonely increasing.

1. FRACTURE CRITERIA OF ANISOTROPIC PLATES WITH CRACKS

In the vicinity of the investigated crack tip A (Fig. 1) we consider the damage function in the process zone at the crack tip $\Pi\left(p, l, b, g\right)$ accepting the crack to be growing in the direction $b = b_*$ of its maximum, at such value of generalized parameter of loading $p = p_*$, when the damage reaches the critical value:

$$\max_{b}\left[\Pi\left(p_*, l, b, g\right)\right] = \left[\Pi\left(p_*, l, b, g\right)\right]\Big|_{b=b_*}, \quad \Pi\left(p_*, l, b_*, g\right) = \Pi_*. \qquad (1)$$

Here b is an angular coordinate of local polar coordinate system Arb with origin A; g is tangent orientation to the axis of the crack at its tip with respect of the main direction of anisotropy Ox with the highest elasticity modulus (Ox, Oh are the principal anisotropy axes). The first equation helps to calculate the initial motion angle b_* while the second calculates the limit loading p_*.

Force measures of damage. Let as introduce the force measures of damage

$$\boldsymbol{p}_b = K_b / K_{Ic}, \boldsymbol{p}_{rb} = K_{rb} / K_{IIc}; \quad (2)$$

$$K_b = \lim_{r \to 0}\left(\sqrt{2\boldsymbol{p} r}\boldsymbol{s}_{bb}\right), K_{rb} = \lim_{r \to 0}\left(\sqrt{2\boldsymbol{p} r}\boldsymbol{s}_{rb}\right); \quad (3)$$

K_{Ic}, K_{IIc} are fracture toughnesses for mode-I and mode-II fracture.

By analogy with Mohr's criterion, which defines fracture initiation on the plane $\boldsymbol{b} = \text{const}$ by the condition $f(\boldsymbol{s}_{bb}, \boldsymbol{s}_{rb}) = 0$, we shall present the damage function Π with the account of the possible effect of additional parameters $\boldsymbol{h}_i$, which characterize the effect of nonsingular factors upon the fracture resistance in the form of the function $\Pi = \Pi(\boldsymbol{p}_b, \boldsymbol{p}_{rb}, \boldsymbol{h}_i)$. Neglecting the parameter $\boldsymbol{h}_i$ and presenting $\Pi = \Pi(\boldsymbol{p}_b, \boldsymbol{p}_{rb})$ in the form $\Pi = \boldsymbol{p}_b{}^4 + \boldsymbol{p}_{rb}{}^4$ at $\boldsymbol{b}_* = 0$ and $\Pi_* = 1$ we shall obtain the Panasyuk – Andreikiv fracture criterion: $K_1^4 / K_{Ic}^4 + K_2^4 / K_{IIc}^4 = 1$.

Neglecting tangent stresses (the mode-I fracture - by normal separation) in $\Pi = \Pi(\boldsymbol{p}_b, \boldsymbol{p}_{rb}, \boldsymbol{h}_i)$ we shall obtain $\Pi = \Pi(\boldsymbol{p}_b, \boldsymbol{h}_i)$. Now, considering $\Pi = \boldsymbol{p}_b{}^4 + \boldsymbol{h}_i{}^2$, $\boldsymbol{h}_i = \boldsymbol{s}_{bb} / \boldsymbol{s}_{0,2}$, $\Pi_* = 1$ we shall get another criterion of the above mentioned scientists: $K_I^4 / K_{Ic}^4 + \boldsymbol{s}_{bb}^2 / \boldsymbol{s}_{0,2}^2 = 1$. If now we neglect parameters $\boldsymbol{h}_i$ for the force damage function, the function of mode-I fracture $\Pi = \Pi(\boldsymbol{p}_b, \boldsymbol{h}_i)$ in the simplest linear case of dependence on $\boldsymbol{p}_b$ may be identified with the first force damage measure: $\Pi \equiv \boldsymbol{p}_b$.

Criterion (1) with the force damage function

$$\Pi(p, l, \boldsymbol{b}, \boldsymbol{g}) \equiv K_b(p, l, \boldsymbol{b}, \boldsymbol{g}) K_{Ic}^{-1}(\boldsymbol{b} + \boldsymbol{g}), \ \Pi_* = 1. \quad (4)$$

generalize in this case $\boldsymbol{s}_{qq}$-criterion, in which the notion of the intensity factor of circumferential hoop stress K_b is also used. Here $K_{Ic}(\boldsymbol{b} + \boldsymbol{g})$ is critical value of the intensity factor K_b where the nonfatigue crack growth is initiated and which depends on the motion direction of the mode-I crack with respect to principal axes $\boldsymbol{x}, \boldsymbol{h}$. As far as $K_b \le K_{Ic}$, then $0 \le \Pi \le \Pi_* = 1$.

If $K_{Ic}(\boldsymbol{b} + \boldsymbol{g}) = \text{const}$ (it is always so for isotropic materials), the condition (1) with measure (4) provides classic relationship for isotropic material:

$$\max_b \left[K_b(p_*, l, \boldsymbol{b}, \boldsymbol{g})\right] = \left[K_b(p_*, l, \boldsymbol{b}, \boldsymbol{g})\right]\Big|_{b = b^*}, \quad K_b(p_*, l, \boldsymbol{b}_*, \boldsymbol{g}) = K_{Ic}. \quad (5)$$

If we consider $K_{Ic} = K_{Ic}(\boldsymbol{b} + \boldsymbol{g})$ in (5), the first equations in (1), (4) and (5) will show different values of $\boldsymbol{b}_*$ when the anisotropy is sufficiently high. Though the second equations in (1), (4) and (5) are identical, the calculated by them limit loadings p_* will vary due to the differences in calculations of $\boldsymbol{b}_*$.

Strain damage function. Generalizing $\boldsymbol{d}_c$-model and the concentration of the limit crack tip opening displacement and linking fracture with plastic strain we shall use the following determination of the measure of material damage at the tip crack:

$$\Pi\left(p, l, \boldsymbol{b}, \boldsymbol{g}\right) \equiv 2v_b\left(p, l, L, \boldsymbol{b}, \boldsymbol{g}\right)\boldsymbol{d}_{Ic}^{-1}\left(\boldsymbol{b}+\boldsymbol{g}\right),\ \Pi_* = 1. \qquad (6)$$

Here $2v_b\left(p, l, L, \boldsymbol{b}, \boldsymbol{g}\right)$ is a normal vector component of possible plastic branch opening (plastic zone) at angle $\boldsymbol{b}$ with crack plane; L is characteristic linear size of the plastic branch which is determined under finity stress condition at its tip; $\boldsymbol{d}_{Ic}\left(\boldsymbol{b}+\boldsymbol{g}\right)$ is experimentally obtained critical COD $\boldsymbol{d}_{Ic}$. After achieving this COD, according to the Leonov – Panasyuk criterion the mode-I crack length change begins. For anisotropic materials the critical COD must depend on the mode-I crack orientation with respect to main anisotropic material axes.

2. TENSION OF ANISOTROPIC PLATE WITH A RECTILINEAR CRACK

When the thin-walled element of the construction is affected only by the tension loading $\boldsymbol{s}_{11}^{\infty} = \boldsymbol{s}_1, \boldsymbol{s}_{22}^{\infty} = \boldsymbol{s}_2$, the intensity of the circumferencial stress factor K_b in the vicinity of the crack tip is determined according to the results of works [3] by the relationship:

$$\begin{gathered}
\operatorname{Re}\Big\{\left(s_1 - s_2\right)^{-1}\Big[-\left(\cos \boldsymbol{b} + s_1 \sin \boldsymbol{b}\right)^{3/2}\left(s_2 K_I + K_{II}\right) + \\
+\left(\cos \boldsymbol{b} + s_2 \sin \boldsymbol{b}\right)^{3/2}\left(s_1 K_I + K_{II}\right)\Big]\Big\} = K_{Ic}\left(\boldsymbol{b}\right), \\
\frac{\partial}{\partial \boldsymbol{b}}\operatorname{Re}\Big\{\left[K_{Ic}\left(\boldsymbol{b}\right)\left(s_1 - s_2\right)\right]^{-1}\Big[-\left(\cos \boldsymbol{b} + s_1 \sin \boldsymbol{b}\right)^{3/2}\left(s_2 K_I + K_{II}\right) + \\
+\left(\cos \boldsymbol{b} + s_2 \sin \boldsymbol{b}\right)^{3/2}\left(s_1 K_I + K_{II}\right)\Big]\Big\} = 0,
\end{gathered} \qquad (7)$$

where $K_I = \sqrt{2\boldsymbol{p}l}\left[\boldsymbol{s}_1 \sin^2 \boldsymbol{a} + \boldsymbol{s}_2 \cos^2 \boldsymbol{a}\right],\ K_{II} = \sqrt{\boldsymbol{p}l/2}\left(\boldsymbol{s}_1 - \boldsymbol{s}_2\right)\sin 2\boldsymbol{a}$ are the stress intensity factors at the crack tip, corresponding to local symmetric and antisymmetric stress condition; $\boldsymbol{a}$ is the effect angle of the principal stresses $\boldsymbol{s}_1$ with respect to crack orientation axis Ox. The material anisotropy is characterized by the complex parameters s_1, s_2, which are the roots

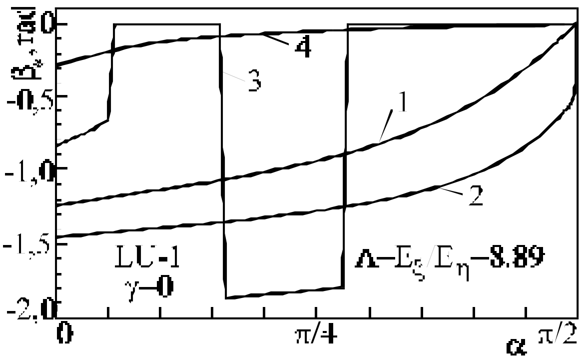

Figure 2: The initial crack motion angle b_* vs. crack orientation angle a $(g = 0)$.

of the characteristic equation of the corresponding elasticity theory problem [4]. From $b = b_*$ roots of the second equation (7) the one, which assigns the maximum value to the left part of the first equation is chosen.

For specific cases of the plate material equations (7) were solved by numerical methods. We considered the cases of tension of a boundless orthotropic plate with l long crack, placed at the infinite points by an uniaxial uniform stress field $s_{22}^{\infty} = s_2 \square p, s_{11}^{\infty} = s_1 = 0$ at angle a with the crack axis, which in its turn, is inclined with respect to the principal orthotropy axis at angle g. Figures 2, 3 shows the $b_* \sim a$ and $p_*^0 \equiv p_* \sqrt{l_*} / K_{Ic}^{\text{avg}} \sim a$ dependences for the transversal isotropic band carbon plastic LU-1 with a medium orthotropy measure $A \equiv E_x / E_h = 8.89$.

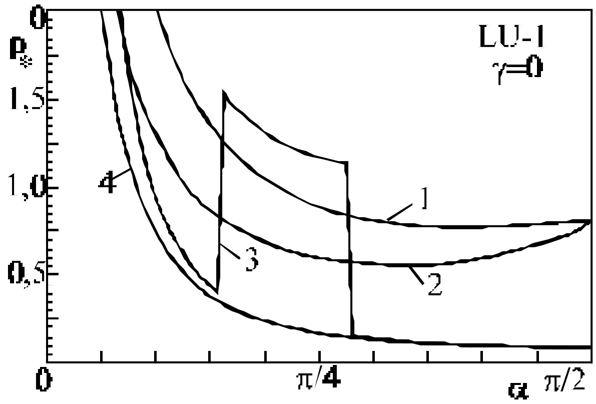

Figure 3: The dependence of the dimentionless critical stress p_*^0 on crack orientation angle a $(g = 0)$.

Here s_{2*} is the critical value of s_2; $K_{Ic}^{\text{avg}} = 250.5$ Nmm$^{3/2}$ is the average value of $K_{Ic}(b)$; $K_{Ic}\big|_{g=0,\, b=0} = 28\text{Nmm}^{3/2}$; $K_{Ic}\big|_{g=0,\, b=p/2} = 473\text{Nmm}^{3/2}$.

Here lines *1* correspond to the numeric solution of equations (7) for isotropic material; lines *2* are obtained for anisotropic material with assumption that $K_{Ic}(b) = \text{const} = K_{Ic}^{\text{avg}}$; lines *3* refer to linear approximation of experimental data [2]; lines *4* refer to physically more real smooth approximation of the some experimental data.

Conclusions. The following main conclusions can be drawn out from the performed calculations:

1. An account of the material anisotropy sufficiently affects the predicted value of the angle β_* and affects less the dimensionless limit loading. A motion angle sign is the same as in the isotropic material – from the closer direction of tension.

2. An account of angular relationship K_{Ic} affects the predicted crack initial motion angle value. For the angle values close to $\beta+\gamma = k\pi/2$ $(k = 0, \pm 1, \pm 2, ...)$ the smooth approximation (lines *4*) provides more significant results.

3. Approximation K_{Ic} by its average value K_{Ic}^{avg} (such situation corresponds to the use of $\sigma_{\theta\theta}$ - criterion) gives values close enough or even identical to the results of the calculations for isotropic material.

4. An account of K_{Ic} dependence on angular coordinate affects essentially the predicted values of the crack initial motion angle. In case the crack is oriented along the principal axis of minimal K_{Ic} (maximal elasticity modulus), these values are lower than the ones calculated for isotropy cases or K_{Ic}^{avg} ($\sigma_{\theta\theta}$-criterion). In case of crack orientation along the principal axis of maximum K_{Ic} (minimal elasticity modulus), these values are higher than the ones calculated in isotropy cases or K_{Ic}^{avg}. The exceptions are angles α close to $\pi/2$ at which in case $\gamma = \pi/2$ the β_* value reaches considerably high positive values.

5. In general, the usage of linear approximation is not only physically incorrect and inconvenient in the analytical transformations but in all cases it gives improbable results for close to mode-I ($\alpha \approx \pi/2$) cracks. On the other hand, linear approximation gives stepwise results for β_* angle at the crack orientation along the maximal elasticity modulus axis. The reason is that criterial function $F_b(\beta,\gamma)$ due to $K_{Ic}(\beta)$ function peculiarities has several local maxima and during the continuous change of β_* angle the location changes continuously as well as their values. Therefore, local maxima, located in other places, may play the role of global maxima in case of different β. Such effect manifested itself only in the case of linear approximation, however, at the higher measure of material anisotropy. The same effect may be expected in the case of smooth approximation when the actual K_{Ic} distribution tends to the linear one with high gradient. The calculations performed for the low anisotropy materials EF 32-301 ($A=1.56$) and ETF ($A=2.37$) did not reveal the continuous change of β_* even when linear approximation had been applied. So, in the case of high material anisotropy the unstable change of angle β_* may be expected.

6. Limit stresses change continuously and are of minor difference at any way (linear or smooth) approximation of angular dependency $K_{Ic}(\boldsymbol{b})$.

7. At $\boldsymbol{a} \to 0$ the calculations give the predicted result $p_*^0 \to \infty$.

8. At $\boldsymbol{a} \to \boldsymbol{p}/2$ the calculations by anisotropy models and K_{Ic}^{avg} give quiet equal values of p_*^0. An account of angular relationship for the material of high anisotropy measure provides the decreased strength.

3. BENDING OF ANISOTROPIC PLATE WITH A RECTILINEAR CRACK

The second case applies to the cracked plate bending by bending moments m_1, m_2 in mutually perpendicular planes. The crack is oriented along axis Ox, which forms angle $\boldsymbol{g}$ with principal orthotropy axis Ox and angle $\boldsymbol{a}$ with the plane of moment action m_2 (see Fig. 1). On the superposition principle the problem may be split into two ones: the problem of the stress-strained state of the same plate without crack under applied to it loading and the problem of the state disturbed by a crack, when the bending intensity moments $-m$, opposite in sign to those, obtained on the crack line in the first problem, are applied to the crack edges. Then the system of equations (1), (4) will have the following form:

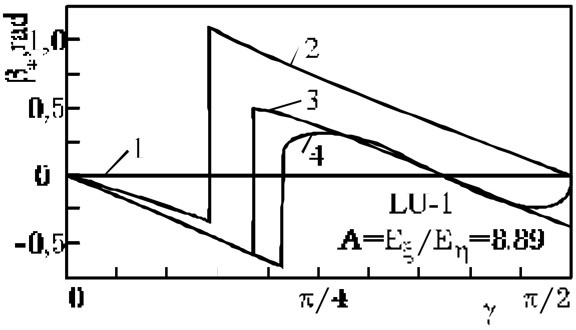

Figure 4: The dependence of initial crack angle motion on orientation of principal anisotropy axes.

$$-K_I \frac{z}{h} \mathrm{Re}\left[\frac{1}{\nabla}\left(p_2 s_1 \frac{T_5}{T_1} - p_1 s_2 \frac{T_6}{T_2}\right)\right] - K_{II} \frac{z}{h} \mathrm{Re}\left[\frac{s_1 s_2}{\nabla}\left(q_2 \frac{T_5}{T_1} - q_1 \frac{T_6}{T_2}\right)\right] = K_{Ic}(\boldsymbol{b}),$$

$$K_I \frac{z}{h} \frac{\partial}{\partial \boldsymbol{b}} \mathrm{Re}\left[\frac{1}{K_{Ic}(\boldsymbol{b})\nabla}\left(p_2 s_1 \frac{T_5}{T_1} - p_1 s_2 \frac{T_6}{T_2}\right)\right] + \qquad (8)$$

$$+ K_{II} \frac{z}{h} \frac{\partial}{\partial \boldsymbol{b}} \mathrm{Re}\left[\frac{s_1 s_2}{K_{Ic}(\boldsymbol{b})\nabla}\left(q_2 \frac{T_5}{T_1} - q_1 \frac{T_6}{T_2}\right)\right] = 0.$$

Here z is a coordinate along the plate thickness $(-h \le z \le h)$; s_j, p_j, q_j, ∇ are known complex parameters [3, 4]; $T_i = T_i(b)$ are known functions [3] of angle b; $K_I = 1.5mh^{-2}\sqrt{pl/2}$, $K_{II} = -CK_I$; C is known real constant [3], which is defined by the complex parameters s_j, p_j, q_j, ∇; $m = m_1 f_1(a)$, $f_1(a) = \sin^2 a + h\cos^2 a$, $h = m_2 / m_1$.

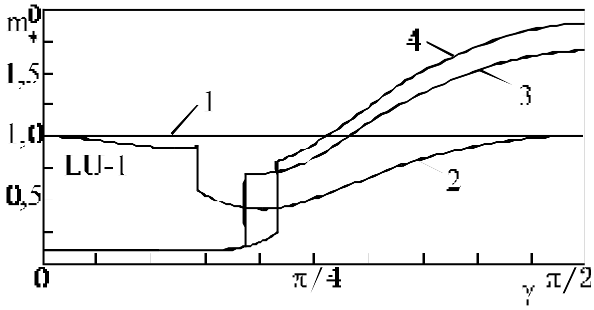

Figure 5: The dependence of dimensionless limit moment m_*^0 on angle g.

For the concrete data of mechanical properties of the cracked plate material and its loading by moments m_1, m_2 equations (8) was solved by numerical methods. Thus, if $m_2 = 0$, the limit values $m_1 = m_{1*}$ of external moment m_1, after reaching which the initial crack motion begins, were found as well as the values of its initial motion direction angle $b = b_*$. Figures 4, 5 show the dependence of angle b_* and of the dimensionless basic function of the limit moment $m_* \equiv m_{1*}^0 \big/ m_{1*}^{0\,(isot)} = m_*(b_*, g) \big/ m_*^{(isot)}$ on angle g. Here $m_{1*}^0 \equiv$ $\equiv m_1 \sqrt{l}(2h)^{-2} \big/ K_{Ic}^{\text{ñåð}} = m_*(b_*, g) f_1^{-1}(a)$. Lines *1* refers to the isotropic case; lines *2-4* refer to the anisotropic case if $K_{Ic} = K_{Ic}^{\text{avg}}$ (corresponds to calculation by s_{qq} - criterion) of linear and smooth approximation of experimental data for $K_{Ic}(b)$ in glass-reinforced plastic LU-1.

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