STUDY OF THE PERFORMANCE OF A MOST PROBABLE POINT CORRECTION WHEN APPLIED TO B-PFEM MODELS IN FATIGUE CRACK PROPAGATION

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ABSTRACT

In a previous paper [3] a new model to predict the cumulative distribution function of fatigue life during the crack propagation stage was described. This problem was there considered as a cumulative damage process following the probabilistic approach of Bogdanoff & Kozin [7], assuming a linear approximation for the random variable “fatigue life” and a truncated uniform distribution for the crack length. In a second work [4], two corrections to this initial model were discussed: a second order approximation of the fatigue life and an analytical expression for the probability density distribution of the crack length derived from the ones of the initial and final crack lengths. The obtained results showed a much better performance, especially for high standard deviations. Finally, a different possibility is studied in this paper: the correction of the initial linear model by a correction of the Wu’s type [18]. This is based on the correction of the abscise of the probability distribution function of the fatigue life, by computing its "exact" value for the estimated values of the state random variables at the most probable point of the initial linear approximation. This approach implies a much lower computational cost than the second order approach cited above, but demonstrates unfortunately not to be adequate when applied to our case. This is due to the strong non-linearity induced by the exponential function that appears in the Paris law, that defines the fatigue life of metals. This effect is clearly shown in the different examples, especially in those with not very small variances.

INTRODUCTION

Fatigue is known to be sensitive to many different parameters that rarely may be considered as deterministic. Stochastic variations of the geometry and dimensions, crack length and direction, material properties and load history have a decisive influence on the fatigue phenomenon, inducing important deviations from the mean or characteristic values of the fatigue life when considered as deterministic. Fatigue is therefore recognised as a random process, which only recently has started to be analysed with the tools of the probability theory.

Two different stages are usually considered during the fatigue process: crack nucleation and crack propagation. They are based on different micro structural damage mechanisms and usually treated separately and combined afterwards to get the total fatigue life. Crack nucleation models are based on local strain approaches [11], while crack propagation is based on the concepts of Fracture Mechanics [6].

In this paper as in other previous works [2][3][5][7]-[9][16]-[18] we study the fatigue crack propagation problem from a probabilistic point of view. This study is the continuation of the work previously presented in [3], in which fatigue crack propagation was considered as a cumulative damage problem, discrete in time, using the probabilistic models early developed by Bogdanoff and Kozin [7] (B-models). However, and on the contrary to the results presented in [7], that were obtained from different series of experimental tests, in [3], the basic stochastic properties of the state variables (displacements, stresses, stress intensity factor, etc.) were computed by means of the so-called Probabilistic Finite Element Method (PFEM) and the random properties of the input data (load history, material properties, initial and final crack length, etc.).

In these B-PFEM models, the initial and final crack lengths, the crack propagation angle, the initial location of the crack, the material parameters and the applied loads are considered as possible random variables. The moments of the fatigue life are computed from the ones of the different variables appearing in the Paris-Erdogan law [14] (material parameters, crack length and stress intensity factor (SIF)), for each crack length considered along the crack path until failure. From these moments, the probability transition matrix of the B-model is computed, and using the Markov laws, the Cumulative Distribution Function (CDF) of the fatigue life is finally obtained.

In [3], different pure mode I examples were described, that allowed us to conclude that the maximum error of the estimated fatigue life was lower than 10% in the design region (the lower failure probability region of the CDF). However, in all the different examples there appeared differences of this order. Trying to improve the accuracy of this approach, different corrections were studied. The first analysed in [4] was the extension of the truncation order of the Taylor expansion of the fatigue life from linear to quadratic to compute the variance. A second correction, also studied in [4], consisted of the modification of the probability distribution function (PDF) of the "crack length". Instead of the truncated uniform distribution, considered in [3], the analytical expression for the PDF of the crack length increment was obtained from the assumed truncated uniform distributions for the initial and final crack lengths and the number of damage levels established to construct the B-model. Thank to these two corrections, much better results both for the mean and standard deviation and for the whole CDF specially around the design region were obtained, as was clearly shown in the examples included in [4].

A second alternative of correction is introduced and analysed in this paper. It is based on the most probable point correction earlier introduced by Wu [17][18], but here extended to include exponential functions as the one appearing in most of the fatigue life prediction models during the crack propagation stage (i.e. Paris model). The idea is to check if this method increases the accuracy of the linear model (linear approximation of the variance) keeping low the computational cost of the problem in comparison with the previously commented two other corrections. The obtained results show, however, that this type of correction is not good for rapidly varying functions as the exponential. In fact, as it will be explained, this correction only ensures improved results over one of the two sides (left or right) of the mean, even in simple random functions. From that we conclude that the best approach, at least for the moment, is the use of the appropriate PDF of the variables and a second order approach for the computation of the moments.

In the forthcoming sections, we shall briefly review the way in which the B-model and the two first mentioned corrections are constructed, assuming known the moments of the different random variables. Also, the random variables of the problem relevant to the fatigue life computation are identified, their relationship with the finite element variables characterised and their corresponding moments obtained. Section 3 introduces the approach based on the Wu's work and its extension to our fatigue problem. In section 4 several results for different examples in mode I are shown and compared with the ones obtained in the previous papers.

REVIEW OF B-PFEM CUMULATIVE DAMAGE MODELS

The model here presented is based on the stochastic approach introduced by Bogdanoff and Kozin [7] and Kozin and Bogdanoff [8] to statistically characterise the results obtained from experimental tests for different probabilistic damage phenomena, including fatigue. These, from now on B-models, are based on the theory of Markov chains. Using standard results in Markov chains, it is possible to write

$$\boldsymbol{p}_t = \{p_t(1), \ \ldots, \ p_t(b)\} \tag{1}$$

with the vector defining the probability of reaching a certain damage level 1,2,..,b at a certain time t, $\boldsymbol{p}_0$ the initial distribution of the damage levels at t=0 and $\boldsymbol{P}$ the so-called *probability transition matrix* which, by the first hypothesis, is assumed to be constant along the process. For a one-step B-model $\boldsymbol{P}$ is of the form

$$\boldsymbol{P} = \begin{bmatrix} p_1 & q_1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & p_2 & q_2 & 0 & \ldots & 0 & 0 \\ 0 & 0 & p_3 & q_3 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & p_{b-1} & q_{b-1} \\ 0 & 0 & 0 & 0 & \ldots & 0 & 1 \end{bmatrix} \tag{2}$$

with p_j the probability of remaining in the same state during one DC, and q_j the probability of advancing to the next level, that is from the damage state j to j+1, fulfilling $1 \geq p_j \geq 0, p_j + q_j = 1$.

Therefore, for the construction of the probability transition matrix the previous computation of the mean and variance of the fatigue life for a certain number of damage levels (crack lengths) is needed. These values are computed in this work by means of the Probabilistic Finite Element Method and the so-called perturbation approach [10]. This method establishes an approximation of the moments of a random function depending on other random variables, by the computation of the same moments of the Taylor expansion of that function about the means of the different variables to which it depends.

In our case, the fatigue life has been defined according to the Paris-Erdogan model [13], that is,

$$T = \int_{a_1}^{a_f} \frac{da}{D(\Delta k_{eq})^n} \tag{3}$$

where a_1 and a_f are the initial and final crack lengths respectively; da is the differential crack length along the crack path; D and n are the Paris material and $\Delta k_{eq}(a)$ is the mode I-equivalent stress intensity factor amplitude along the load time history. In order to provide a higher generality, all of these parameters have been initially considered as possible random variables

If we now discretise equation (3) in *ns* crack growth steps, we can approximate it by

$$T=\sum_{i=1}^{ns}\frac{A_i}{D\left(K_{eq_i}\right)^N}=\sum_{i=1}^{ns}T_i \tag{4}$$

where A_i is the increment of crack length during step i, D and N are the Paris parameters[1], K_{eq_i} the mode I-equivalent stress intensity factor amplitude and ns the number of steps in which the crack propagation process is divided. If we now establish the Taylor expansion of T_i up to second order around the means of the different random variables A_i, D, N and K_{eq_i}, we obtain

$$T_i=\frac{A_i}{DK_{eq_i}^N}\approx\frac{\mu_{A_i}}{\mu_D\left(\mu_{K_{eq_i}}\right)^{\mu_N}}+\sum_{j=1}^{4}\left.\frac{\partial T_i}{\partial X_j}\right|_{\mu_{X_j}}\left(X_j-\mu_{X_j}\right)+\frac{1}{2!}\sum_{\substack{j=1\\k=1}}^{4}\left.\frac{\partial^2 T_i}{\partial X_j\partial X_k}\right|_{\substack{\mu_{X_j}\\\mu_{X_k}}}\left(X_j-\mu_{X_j}\right)\left(X_k-\mu_{X_k}\right) \tag{5}$$

where μ_{X_i} is the mean of the corresponding random variable X_i (A_i, D, N or K_{eq_i}), index i varies from 1 to ns meaning the number of steps (damage levels) and the indices j and k vary from 1 to 4 meaning the four random variables considered. The approximate mean of T_i is obtained by applying the expectation operator to (5). After some algebra and using the expressions included in Bea [2], we get

$$E\left[\sum_{i=1}^{ns}T_i\right]=\sum_{i=1}^{ns}\frac{\mu_{A_i}}{\mu_D\left(\mu_{K_{eq_i}}\right)^{\mu_N}}+\frac{1}{2!}\left(2\sum_{i=1}^{ns}\left[\left.\frac{\partial^2 T_i}{\partial A_i\partial K_{eq_i}}\right|_{\mu_{A_i};\mu_{K_{eq_i}}}\left.\frac{\partial K_{eq_i}}{\partial A}\right|_{\mu}Var\left(A_i\right)\right]+\sum_{i=1}^{ns}\sum_{j=1}^{4}\left.\frac{\partial^2 T_i}{\partial X_j^2}\right|_{\mu_{X_j}}Var\left[X_j\right]\right) \tag{6}$$

For the computation of the variance of T_i we will use again (3) but only up to order one, that is

$$\sum_{i=1}^{ns}T_i=\sum_{i=1}^{ns}\frac{A_i}{DK_{eq_i}^N}\approx\sum_{i=1}^{ns}\frac{\mu_{A_i}}{\mu_D\left(\mu_{K_{eq_i}}\right)^{\mu_N}}+\sum_{i=1}^{ns}\sum_{j=1}^{4}\left.\frac{\partial T_i}{\partial X_j}\right|_{\mu_{X_j}}\left(X_j-\mu_{X_j}\right) \tag{7}$$

Considering the independence of the different random variables, the variance of (7) may be written as

$$\begin{aligned}Var\left(\sum_{i=1}^{ns}T_i\right)\approx&\sum_{i=1}^{ns}\left(\left.\frac{\partial T_i}{\partial A_i}\right|_{\mu_{x^i}}\right)^2Var\left(A_i\right)+\sum_{i=1}^{ns}\left(\left.\frac{\partial T_i}{\partial D}\right|_{\mu_{x^i}}\right)^2Var(D)+\sum_{i=1}^{ns}\left(\left.\frac{\partial T_i}{\partial K_{eq_i}}\right|_{\mu_{x^i}}\right)^2Var\left(K_{eq_i}\right)+\\&+\sum_{i=1}^{ns}\left(\left.\frac{\partial T_i}{\partial N}\right|_{\mu_{x^i}}\right)^2Var(N)+\sum_{i=1}^{ns}\sum_{\substack{j=1\\j\neq i}}^{ns}\left.\frac{\partial T_i}{\partial D}\right|_{\mu_{x^i}}\left.\frac{\partial T_j}{\partial D}\right|_{\mu_{x^j}}Var(D)+\sum_{i=1}^{ns}\sum_{\substack{j=1\\j\neq i}}^{ns}\left.\frac{\partial T_i}{\partial N}\right|_{\mu_{x^i}}\left.\frac{\partial T_j}{\partial N}\right|_{\mu_{x^j}}Var(N)+\\&+\sum_{i=1}^{ns}\left.\frac{\partial T_i}{\partial A_i}\right|_{\mu_{x^i}}\left.\frac{\partial T_i}{\partial K_{eq_i}}\right|_{\mu_{x^i}}\left.\frac{\partial K_{eq}}{\partial A}\right|_{\mu_{x^i}}Var\left(A_i\right)+\sum_{i=1}^{ns}\sum_{j=1}^{i}\left.\frac{\partial T_j}{\partial A_j}\right|_{\mu_{x^j}}\left.\frac{\partial T_i}{\partial K_{eq_i}}\right|_{\mu_{x^i}}\left.\frac{\partial K_{eq}}{\partial A}\right|_{\mu_{x^j}}Var\left(A_j\right)\end{aligned} \tag{8}$$

In (6), (8), the moments of the material parameters are assumed to be known, the crack length for each damage state is assumed to be a truncated uniform distribution with known mean defined by the means of the initial and final crack lengths and the number of damage states. Finally, the moments of the stress intensity factor for each damage state (crack length values needed to accurately integrate the Paris law) have to be evaluated.

Following this approach, several examples were checked [3], concluding that the maximum error of the estimated fatigue life was lower than 10% in the lower failure probability region of the CDF (design region). However, in all the different examples there appeared differences of this order and therefore different corrections were studied to improve those results. The first correction considered in [4] was the extension of the truncation order of the Taylor expansion of the fatigue life from linear to quadratic to compute the variance. Again, the approximated variance of the fatigue life is computed by applying the appropriate operator to (5). This variance may be expressed as

$$\begin{aligned}Var[T]=&\frac{1}{4}\sum_{i=1}^{n}\sum_{l=1}^{n}\left(\sum_{j=1}^{4}\left.\frac{\partial^2 T_i}{\partial X_j^2}\right|_{\mu_i}\left.\frac{\partial^2 T_l}{\partial X_j^2}\right|_{\mu_l}\left(E\left[\left(X_j-\mu_{X_j}\right)^4\right]-Var^2\left[X_j\right]\right)\right)+\\&+\sum_{i=1}^{n}\sum_{l=1}^{n}\left(\sum_{\substack{j=1\\k>j}}^{4}\left.\frac{\partial^2 T_i}{\partial X_j\partial X_k}\right|_{\mu_i}\left.\frac{\partial^2 T_l}{\partial X_j\partial X_k}\right|_{\mu_l}Var\left[X_j\right]Var\left[X_j\right]\right)\end{aligned} \tag{9}$$

A second correction, also studied in [4], consists of the modification of the probability distribution function (PDF) of the "crack length". This variable is not independent as assumed in the initial model. On the contrary, this variable depends on the probability distributions of the initial and final crack lengths and the number of steps considered if we assume constant crack length increments.

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[1] In this section, the different parameters are written in caps to make explicit their character of random variables.

If we still consider truncated uniform distributions for the initial and final crack lengths, that is

$$\begin{cases} a_f \rightarrow u(inf_f, sup_f) \\ a_i \rightarrow u(inf_i, sup_i) \end{cases} \tag{10}$$

where *sup* and *inf* denote the upper and lower bounds of each random variable respectively. The first moments for these two variables are straightforward from the basic properties of probability theory, being

$$E[a_k] = \frac{sup_k + inf_k}{2} \qquad\qquad Var[a_k] = \frac{(sup_k - inf_k)^2}{12} \tag{11}$$

With this, and taking into account the expression of the random variable "crack length increment" in terms of the initial and final crack length and the number of steps

$$A = \frac{a_f - a_i}{ns} \tag{12}$$

we can get immediately the probability distribution function for the crack length increment, (see [4]).

THE WU CORRECTION

Y.T. Wu [18] proposed a method initially efficient to compute CDFs of complex random functions, being especially indicated for those cases where the computation of such function implies a high computational cost, becoming, therefore, the Monte Carlo method clearly inefficient. The Wu's scheme suggests a correction of the CDFs obtained with a linear approximation perturbation method. This is our case, at least for the initial model, for the evaluation of the variance of the fatigue life, and the reason of the study here performed.

We now extend the idea of the Wu's correction to the function "fatigue life" in order to correct the results obtained by the use of only a linear approximation of this function to compute its variance. We follow therefore the same steps than above, particularising for our problem. The function we shall consider is the Paris expression of the fatigue life (3) and its associated linear expansion (7) that we rewrite in the following way

$$Z_p = \sum_{i=1}^{n} \frac{\mu_A}{\mu_D \mu_{Keq_i}^{\mu_N}} + \sum_{i=1}^{n} \sum_{j=1}^{4} \left.\frac{\partial T_i}{\partial X_j}\right|_{\mu} \left(X_j - \mu_{X_j}\right) = T_0 + T_A + T_D + T_N + T_P \tag{13}$$

with

$$T_0 = \sum_{i=1}^{n} \frac{\mu_A}{\mu_D \mu_{Keq_i}^{\mu_N}} \qquad T_A = \sum_{i=1}^{n} \left.\frac{\partial T_i}{\partial A}\right|_{\mu} (A - \mu_A) \qquad T_D = \sum_{i=1}^{n} \left.\frac{\partial T_i}{\partial D}\right|_{\mu} (D - \mu_D) \qquad T_N = \sum_{i=1}^{n} \left.\frac{\partial T_i}{\partial N}\right|_{\mu} (N - \mu_N) \qquad T_P = \sum_{i=1}^{n} \left.\frac{\partial T_i}{\partial P}\right|_{\mu} (P - \mu_P) \tag{14}$$

and, we follow the next steps

1. For each value T of the fatigue life used to compute the corresponding CDF we obtain the values of the random variables A, D, N, P associated to the most probable point of the space of random variables constrained to he assumed fatigue life value. These values will be denoted as A_c, D_c, N_c, P_c..
2. We evaluate the expression (4) for these values, getting a new value of the fatigue life denoted by T_c .
3. Finally, we correct the CDF of the fatigue life identifying the value corresponding to T with the one of the corrected function associated to T_c .

Therefore, the only step we have to solve is the first one, that is, the computation of the most probable point for each value of the linear expansion (13). The values of the random variables *A, D, N, P* are those which maximise the joint probability density function f_{ADNP} which, assuming independent random variables is only the product of the four independent PDFs of each of the variables, constrained to the value T of the fatigue life. The problem to solve is therefore formulated as:

$$\left| \begin{array}{ll} \text{maximise} & f_A \cdot f_D \cdot f_N \cdot f_P \\ \text{constrained to} & T = T_0 + T_A + T_D + T_N + T_P \end{array} \right. \tag{15}$$

or

$$\left| \begin{array}{ll} \text{maximise} & f_A \cdot f_D \cdot f_N \cdot f_P \\ \text{constrained to} & A = \dfrac{T - (T_0 + T_D + T_N + T_P)}{\displaystyle\sum_{i=1}^{n} \left.\frac{\partial T_i}{\partial A}\right|_{\mu_i}} + \mu_A \end{array} \right. \tag{16}$$

assuming established the value of A for each damage level. In the following, we consider, as usual, D, N, P as normally distributed, and the PDF of the crack length.

In some cases (this is for instance the situation that occurs in the examples presented in section 4) the obtained values for A_c are outside the allowed domain. In those cases we fix $A_c = \mu_A$ and compute the rest of variables D_c, N_c and P_c maximising f_{DNP} constrained to $T = T_0 + T_A + T_D + T_N + T_P$. With this, the obtained results are the following

$$D_c = \sum_{i=1}^{n} \left.\frac{\partial T_i}{\partial D}\right|_{\mu_i} \sigma_D{}^2 \left(\frac{T - T_0}{\alpha}\right) + \mu_D \qquad N_c = \sum_{i=1}^{n} \left.\frac{\partial T_i}{\partial N}\right|_{\mu_i} \sigma_N{}^2 \left(\frac{z_P - T_0}{\alpha}\right) + \mu_N \qquad P_c = \sum_{i=1}^{n} \left.\frac{\partial T_i}{\partial P}\right|_{\mu_i} \sigma_P{}^2 \left(\frac{z_P - T_0}{\alpha}\right) + \mu_P \qquad A_c = \mu_A \tag{17}$$

with

$$\alpha = \left(\sum_{i=1}^{n} \left.\frac{\partial T_i}{\partial D}\right|_{\mu_i}\right)^2 \sigma_D{}^2 + \left(\sum_{i=1}^{n} \left.\frac{\partial T_i}{\partial N}\right|_{\mu_i}\right)^2 \sigma_N{}^2 + \left(\sum_{i=1}^{n} \left.\frac{\partial T_i}{\partial P}\right|_{\mu_i}\right)^2 \sigma_P{}^2 \tag{18}$$

Once the variables corresponding to the most probable point are computed the rest of the process is straightforward. The next section shows some of the obtained results.

RESULTS

In this section, the same examples of Mode I crack propagation in metals fatigue included in [3][4] are discussed. All of them correspond to the normalised compact tension specimen of the ASTM E647 standard, with width W=50 and thickness h=6.

Mode I implies a deterministic null crack propagation angle. The first example considers the data shown in Table 1, with the terminology included in [3] and units of the International Standard. In this first case, only the parameter N and the initial and final crack lengths (and therefore the crack length increment) have been considered random.

Figs. 1 and 2 show the complete cumulative probability distribution function and some details, obtained after applying the initial and the corrected B-models constructed using 10 different crack lengths (ns=10). These results are compared to the equivalent ones obtained by a standard Monte Carlo simulation, using the explicit expression for the mode I stress intensity factor defined in the E647 ASTM standard with 400.000 samples. As can be seen, and for both models, the agreement is good for engineering purposes with a maximum difference between the assumed “exact” results and the ones corresponding to the initial B-model of about 1.000 cycles. This corresponds to relative errors lower than 10%, while the error of the corrected B-FEM model reduces to a maximum of about 5%.

TABLE 1
RANDOM INPUT VARIABLES FOR EXAMPLE 1

RANDOM VAR.	MEAN	VARIANCE
a_1 (Truncated Uniform)	3.2 E-3	0.333333 E-6
a_f(Truncated Uniform)	21.2 E-3	0.333333 E-6
A (Random)	2.00 E-3	8.230452 E-9
D (Deterministic)	1. 54236 E-33	0.0
N (Gaussian)	3.6000143	9.98575 E-5
P (Deterministic)	5850.0	0.0

Results predicted by the initial B-model are conservative while the ones corresponding to the corrected B-model are not. However, this is not a common feature for all the different examples studied and has to be considered as a special situation for this type of example.

Finally, the shapes of the three distributions are very similar, appearing as translated one respect to each other. In fact, a constant translation of the abscise seems to be able of reducing strongly the computed error in the lower and medium probability parts of the curves and especially for the corrected model. This again seems to recommend a solution of the Wu's type as previously commented.

In the same figure, the curve obtained after applying the Wu’s correction to the linear corrected model (corrected PDF of the crack length variable but linear approximation of the variance) is also shown. As the others, the result is very accurate being clearly better than the initial model and very close to the corrected model which makes us to wonder if the obtained improvement is essentially due to the first correction, that is, the proper definition of the PDF of the crack length. This is clarified in the next example.

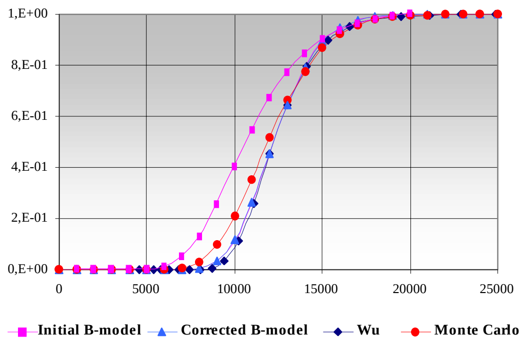

Figure 1. Comparison between the Monte Carlo simulation and the B-model results in example 1

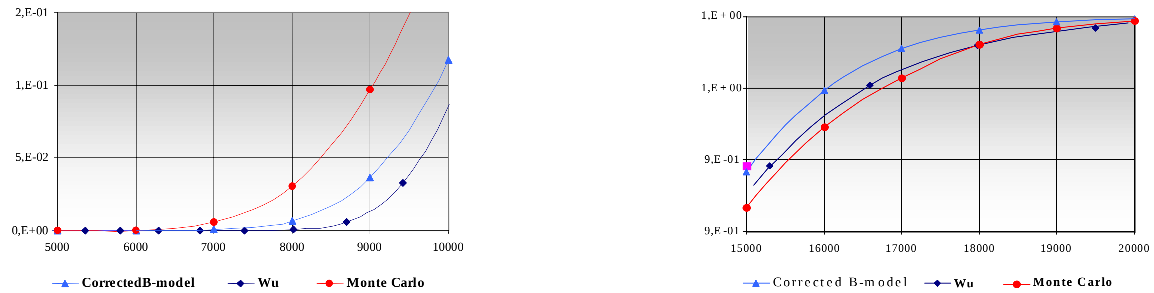

Figure 2. Some details (lower and upper parts) of the Fig. 1

In this first example, it is very difficult to distinguish between the improvement due to the modification of the PDF of the crack length and the one associated to the increment of the truncation order for the computation of the variance. In fact, if we use a first order approximation of the variance but keeping the corrected PDF of the crack length, the obtained curve is indistinguishable from the corrected one in Fig. 1. This is not surprising since a linear approximation of the variance is enough to get a good accuracy when the standard deviations of the random variables are small.

This good accuracy of the initial model is broken when higher variances for the basic random variables are considered. This may be seen in Fig. 3 which corresponds to the cumulative distribution functions computed by the same three models (initial B-model, corrected B-model and Monte Carlo simulation) for the same problem but with the random variables defined in Table 2. In that figure the CDF corresponding to the model with only the first correction included, that is linear approximation of the variance and corrected PDF of the crack length are also shown.

TABLE 2

RANDOM INPUT VARIABLES FOR EXAMPLE 2

RANDOM VAR.	MEAN	VARIANCE
a_i (Truncated Uniform)	3.2 e-3	0.333333 e-6
a_f(Truncated Uniform)	21.2 e-3	0.333333 e-6
A (Random)	2.00e-3	8.230452e-9
D (Deterministic)	1. 54236 e-33	0.0
N (Gaussian)	3.6000143	1.99203e-3
P (Deterministic)	5850.0	0.0

As it can be noticed, the variance of the Paris parameter N is much higher and the accuracy of the initial B-model decreases strongly. This is due not only to the inability of the linear approximation of the variance to approximate the behaviour of the curve far from the mean well enough, but also to the poor approximation of the mean due to the bad crack length PDF employed. This effect also appears in the corrected B-model but very much reduced, specially in the lower part of the CDF (the one with lower failure probabilities) which usually corresponds to the design region and therefore the most interesting for the designer. The second order approximation of the variance only gives, in this problem, a small additional improvement. Of course, this aspect becomes more important for higher variances.

With respect to the Wu correction, the corresponding CDF is also compared in Fig. 4 and the details of the same curves shown in Fig. 5, with the initial, the corrected B-model and the Monte Carlo simulation. From these results we can deduce similar conclusions than the ones cited for example 1, that is, the Wu correction implies an important improvement with respect to the initial model. However, this is due to the better specification of the PDF of the crack length, not having a significant improvement when compare not only to the complete corrected model but also to the linear model with correction on the crack length PDF. This is especially true in the lower part of the curve as can be seen in Fig. 5a, where we are more interested. This behaviour also appears when correcting simpler random functions, so we can conclude that the Wu's correction is useful only for the upper region of the random variable. Specifically for our case, it is not appropriate mainly due to the strong non-linearity of the fatigue life function in terms of the different parameters and, in particular, of the exponent N, which is the most important variable in this example.

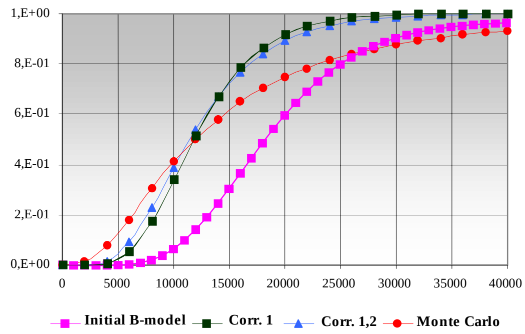

Figure 3. Comparison between the Monte Carlo simulation and the B-model results in example 2

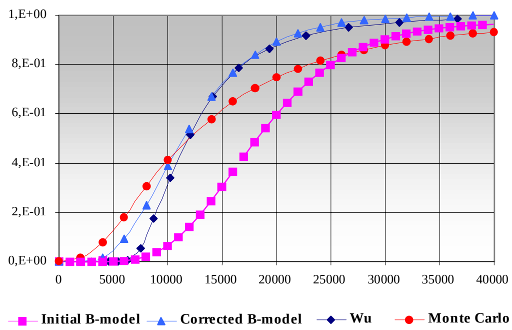

Figure 4. Comparison between the Monte Carlo simulation, the B-model and the Wu correction results in example 2

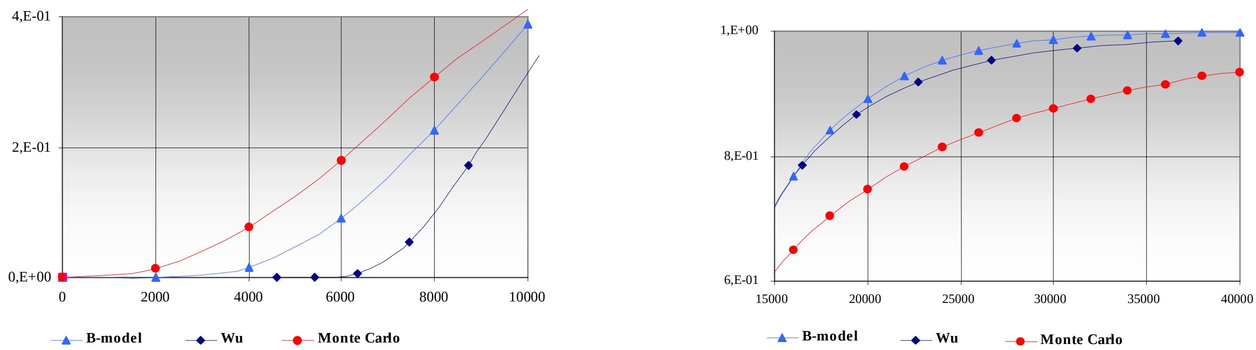

Figure 5. Some details (lower and upper parts) of the Fig. 4

CONCLUSIONS

In the present work the probabilistic approach earlier introduced in Bea et al [2][3][4], based on the application of the Bogdanoff & Kozin B-models for cumulative damage to the fatigue crack propagation problem in metals has been briefly reviewed. This method allows us to compute the cumulative distribution function of the fatigue life during the crack propagation stage. The transition probability matrix of the B-model is constructed from the application of the expectation and variance operators to the Taylor expansion of the Paris-Erdogan fatigue life expression, while the moments of the different random variables are computed from the corresponding input data, from the assumed probability distributions of the initial and final crack lengths and from the results of a set of probabilistic finite element problems for each of the initially established damage (crack lengths) levels. The initial crack length is usually identified with the threshold of the accuracy of the available crack detection equipment, while the final crack length is considered as the established failure.

Besides the external load, the Paris-Erdogan material parameters *D* and *n*, the initial and final crack lengths, the initial crack propagation angle, and the initial location of the crack have been considered as possible random variables. Finally, the elastic material parameters may also be stochastic as some of the geometric properties (thickness of the specimen) that are treated inside the PFE modulus. All of these variables have been considered as Gaussian, although the extension to other situations is straightforward with no more that using a change of variables from the assumed probability distribution to the equivalent Gaussian (see Papoulis [12], for instance).

The accuracy of the obtained failure probability distributions proves a good performance, when compare to an equivalent analysis performed with a Monte Carlo simulation approach or with the results obtained from experimental tests (see Bea et al [3]). This is especially true for problems with small variances (relative errors of about 5-10% for the fatigue life). However, for higher variances some corrections have to be included. Some of them have been already discussed in [4].

Other possibility has been introduced here, the application of a correction to the abscise in a similar fashion to the one proposed by Wu [18], reducing importantly the associated computational cost when compared to other possible improvements. This correction, although leads to a good behaviour for the final region of the CDF (above the mean), provides not very good results in the design region (low failure probabilities), implying that it should be discarded for design purposes in our problem. This is probably due to the strong non-linearity associated to the fatigue life Paris expression, especially in relation to the exponential part of the function.

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