PLANE BOUNDARY VALUE PROBLEM POSED ON ORIENTATION OF PRINCIPAL STRESSES ON THE CRACK SURFACE

A. N. Galybin

Department of Civil and Resource Engineering,
the University of Western Australia, Nedlands, 6907, Australia

ABSTRACT

The paper suggests an approach for consideration of the problem for cracks with contacting surfaces. The approach is based on posing the boundary conditions in terms of principle stress orientations on the crack surface. The magnitudes of stresses remain unknown.

This boundary value problem is considered for an infinite plane with a straight crack. The method of singular integral equations is applied to analyse the solvability of the problem. It is shown that the solvability depends upon the number of rotations of the principal stress axes while traversing the crack. In contrast to the classical formulations (where the problem has unique solution) the number of rotations determines the number of linearly independent solutions of the problem. Thus, a general solution containing a certain number of unknown constants can be built.

INTRODUCTION

Common formulations of the boundary value problem for a 2D crack require assigning normal and shear stresses (or their combinations) on the crack surface. In most cases of cracks propagating under tensile stresses the crack surfaces do not interact with each other. Hence the formulation of the boundary conditions does not meet any difficulties, solution of the problem is unique and consistent with the mechanical point of view. On the other hand when a body in subjected to compressive or shear loading the crack surfaces may be in full or partial contact. The latter substantially complicates formulation of the boundary value problem because the positions of contact zones are *a-priory* unknown. Hence some hypotheses regarding configuration of contact zones should be used, which can lead to paradoxical results. The contact problem for a stamp indented into the boundary of a half-plane without sliding illustrates this statement. The exact solution shows oscillation of the contact stresses near the stamp edges, which contradicts with the assumption that the stamp is in contact everywhere. In this example the size of the oscillation zone (where solution is not valid) is comparatively small and the effect is often neglected. However, this effect should be taken into account if one intends to investigate the stress distribution in the half-plane under the stamp edge. This is the case of interest for Fracture Mechanics since cracks start to grow from the region situated under the stamp edges. Thus, a correction of the boundary conditions may be required to provide consistence of obtained results. However this presumes that some new assumptions will be needed for describing the contact conditions.

The present paper suggests an alternative approach to consider the problem for cracks with contacting surfaces. The approach is based on posing the boundary conditions in terms of principle stress orientations on the crack surface. Galybin and Mukhamediev [1] established the solvability of this problem for the case of a 2-D domain bounded by a smooth closed contour. Here this type of boundary condition is investigated for an open contour (straight crack) in the infinite plane. The information on the stress orientation can be determined in lab experiments. Also there are (e.g. [2]) well-developed methods of the determination of stress orientations in the earth's crust near tectonic faults which can be modelled by shear (or dilating shear) cracks.

It should be emphasised that the knowledge of stress/displacement surface magnitudes is not required in the approach proposed. As it has been shown in [1], this type of boundary conditions may lead to the loss of uniqueness of the solution depending upon the number of rotations of the principal stress axes while traversing the contour. In contrast to the classical formulations (where the problem has a unique solution) the number of rotations determines the number of linearly independent solutions of the problem. Thus a general solution containing a certain number of unknown constants can be built. It is suggested to find these constants by using a number of local stress or displacement measurements.

BASIC RELATIONSHIPS AND BOUNDARY CONDITIONS

A plane problem of elasticity can be formulated by means of the stress functions *P*, *D*. These function are related to the stress components σ_x, σ_y, σ_{xy} in Cartesian coordinates *Oxy* as follows

$$P(x,y)=\frac{1}{2}\left[\sigma_x(x,y)+\sigma_y(x,y)\right],\quad D(x,y)=\frac{1}{2}\left[\sigma_y(x,y)-\sigma_x(x,y)\right]+i\sigma_{xy}(x,y) \tag{1}$$

By introducing the complex variable *z* and

$$z=x+iy,\quad \bar{z}=x-iy \tag{2}$$

one can write the Kolosov formulae (e.g., [3]) for the general solution of a plane elastic problem in the form

$$P(z,\bar{z})=\Phi(z)+\overline{\Phi(z)},\quad D(z,\bar{z})=\bar{z}\Phi'(z)+\Psi(z) \tag{3}$$

where Φ(z), Ψ(z) are holomorphic functions of the complex variable *z*. Here the same notations for the functions *P* and *D* are kept, although after the substitution of complex variables from Eqn. 2 they becomes different. Nevertheless for any functions of two variables *f* one can keep in mind that $f=f(x,y)=$ $=f\left(\frac{z+\bar{z}}{2},\frac{z-\bar{z}}{2i}\right)=f(z,\bar{z})$. As can be seen from Eqn. 3 the function *P* is a harmonic function while the function *D* is a bianalytic function inside a domain.

The major, T_1, and minor, T_2, principal stresses can be determined by the stress functions *P* and *D* as follows

$$T_1=P+|D|,\quad T_2=P-|D|,\qquad T_1\geq T_2 \tag{4}$$

Here and further throughout the text the arguments $z,\bar{z}$ of any complex-valued function may be omitted if formulae are valid everywhere in the region. On the boundary the argument will be shown if necessary.

It is convenient to represent the bianalytic function *D* in the complex exponential form

$$D=|D|e^{i\arg D},\quad |D|=\frac{1}{2}(T_1-T_2) \tag{5}$$

The argument of *D* can also be associated with the angle (φ) of inclination of the major principal stress T_1 with respect to the *x*-axis as $\arg D=-2\varphi$.

Let the Cartesian coordinates *Oxy* be placed in the middle of a straight crack such that the crack occupies the interval $(-l, l)$ on the *x*-axis and is orthogonal to the *y*-axis.

In [1] it has been shown that if the orientation of principal stresses and curvature of their trajectories at the crack boundary are used as the boundary conditions then the problem can be reduced to determination of the function D by the following conditions

$$\arg D = \alpha(s), \quad \left(\arg D\right)'_n = \alpha'_n(s) \tag{6}$$

Here n is the outward unit normal to the contour; $\alpha(s)$ and $\alpha'_n(s)$ are given functions on the contour. In the case considered here s can be associated with x and the derivative with respect to the outward normal can be associated with the derivative with respect to y. In the complex variables this operation takes the form

$$f'_y(x,y) = i\left(\partial f'_z(z,\bar{z}) - \partial f'_{\bar{z}}(z,\bar{z})\right) \tag{7}$$

The boundary values of an arbitrary complex-valued function of two variables can be specified as

$$f^{\pm}(t) = \lim_{z \to t \pm i0} f(z,\bar{z}) \tag{8}$$

where $t \in (-l, l)$ and indices "±" are the boundary values for the upper and lower bound of the crack correspondingly when traversing the contour in the positive direction.

In contrast to the case of closed contours, here a boundary condition at infinity should also be specified. This condition could be posed in stress orientations as well. However for the sake of simplicity it is assumed that the stress functions vanish at infinity.

$$P \to 0, \quad D \to 0, \quad |z| \to \infty \tag{9}$$

Thus from the mechanical point of view the situation corresponds to the case of a plane with the crack which surfaces are loaded by unknown tractions. Body forces and stresses at infinity are absent.

According to [1] a pair of boundary conditions specified by Eqn 6 can be presented in the form of a single complex boundary condition. For the case considered it takes the form

$$\operatorname{Im}\left(e^{-i\alpha(t)}D(t)\right) = 0, \quad \operatorname{Im}\left(e^{-i\alpha(t)}D'_y(t)\right) = e^{-i\alpha(t)}\alpha'_y(t)D(t), \quad t \in (-l, l) \tag{10}$$

Here the boundary values of $D'_y(t)$ and $\alpha'_y(t)$ can be different on upper and lower surface of the crack.

INTEGRAL EQUATIONS

From Eqn 9 it follows that the holomorphic functions $\Phi(z)$, $\Psi(z)$ can be presented by the Cauchy integrals. Following [4] the representation can be made in the form

$$\Phi(z) = \frac{1}{2\pi i}\int_{-l}^{l} \frac{g(t)}{t-z}dt, \quad \Psi(z) = \frac{-1}{2\pi i}\int_{-l}^{l}\left[\frac{\overline{g(t)}}{t-z} + \frac{tg(t)}{(t-z)^2}\right]dt \tag{11}$$

where the function $g(t)$ is proportional to the derivative of the displacement jump across the crack contour

$$g(x) = u(x) + iv(x) = \frac{2G}{1+\kappa}\left(u_x(x,0+) - u_x(x,0-) + iu_y(x,0+) - iu_y(x,0-)\right)'_x \tag{12}$$

where G is shear modulus, $\kappa=3-4\nu$ for the plane strain assumption and $\kappa=(3-\nu)(1+\nu)^{-1}$ for the plane stress, ν in Poison's ratio.

With the account for Eqn, 3 and Eqn. 13 one has the following representation for the function D

$$D = \frac{-1}{2\pi i}\int_{-l}^{l}\left[\frac{\overline{g(t)}}{t-z} + \frac{(t-\bar{z})g(t)}{(t-z)^2}\right]dt \tag{13}$$

Applying Eqn 7, for the normal derivative one can derive the following expression

$$D'_y = \frac{-1}{2\pi}\int_{-l}^{l}\left[\frac{3g(t)+\overline{g(t)}}{(t-z)^2} + \frac{2(z-\bar{z})g(t)}{(t-z)^3}\right]dt \tag{14}$$

Boundary values of D can be obtained by direct applying the Sokhotski-Plemelj formulae to Eqn 13. Their application is obvious for the first integrand and as shown in [4] still valid for the second integrand.

$$D^{\pm} = \mp u + i\mathbf{I}(u) \tag{15}$$

where $u(t)=\mathrm{Re}(g(t))$ and $\mathbf{I}u$ is the singular operator defined as

$$\mathbf{I}(u) = \frac{1}{\pi}\int_{-l}^{l}\frac{u(t)}{t-x}dt \tag{16}$$

Determination of the boundary values for D'_y are not so obvious due to the presence of singularities of higher order in Eqn 14. Since the function $g(t)$ is unbounded at the ends, this restricts the direct integration by parts which is required to obtain the boundary values. However as it is shown in the Appendix, the Sokhotski-Plemelj formulae remain valid in their standard form, which leads to

$$D'^{\pm}_y = \mp i\left(g' + u'\right) - \mathbf{I}\left(g' + u'\right) \tag{17}$$

Here and further on functions marked with prime without the sub-index means that they are differentiated with respect to x.

Substituting these boundary values into Eqn 10 one arrives to the system of four singular integral equations

$$\pm u\sin\alpha^{\pm} + \cos\alpha^{\pm}\mathbf{I}u = 0, \quad \sin\alpha^{\pm}\left[\mp v' + 2\mathbf{I}(u') - \alpha'^{\pm}\mathbf{I}(u)\right] + \cos\alpha^{\pm}\left[\mp 2u' - \mathbf{I}(v') \pm \alpha'^{\pm}u\right] = 0 \tag{18}$$

The first two equation of this system are solvable only if the functions $\alpha^{\pm}$ depend on each other. This dependence has the form

$$\sin\left(\alpha^{+} + \alpha^{-}\right) = 0 \quad \Rightarrow \quad \alpha^{+} + \alpha^{-} = \pi k, \quad k = 0, \pm 1 \ldots \tag{19}$$

By denoting $\alpha=\alpha^{+}$ and accounting for $\exp(i\alpha^{+})=\exp(i\alpha)$ and $\exp(i\alpha^{-})=(-1)^k\exp(-i\alpha)$, one can find that for the solvability of the last two equations of the system the following necessary conditions should be imposed on the boundary values of normal derivatives of the argument of D

$$\alpha'^{+}_y + \alpha'^{-}_y = 0 \tag{20}$$

Let the term “principal stress trajectories” denote the curves, tangents to which coincide with the directions of the corresponding principal stress T_1 or T_2 at any point. Principal stress trajectories form a curvilinear orthogonal net inside the domain. Then the formulae given by Eqns 19-20 express *the refraction law of the trajectories of principal stresses* on the loaded crack in the plane.

Then the system of for equations reduces to the following system of two equations

$$u\sin\alpha+\cos\alpha\mathbf{I}u=0,\quad \sin\alpha\left[v'+2\mathbf{I}(u')+\alpha'_y\mathbf{I}(u)\right]+\cos\alpha\left[\mathbf{I}(v')-2u'-\alpha'_y u\right]=0 \tag{21}$$

where $\alpha'_y=\alpha'^{+}_y$.

Both equations of this system represent singular integral equations of the dominant type. Since the first equation in the system does not depend on v, the equations can be solved separately by reducing them to corresponding Riemann boundary value problems, Gakhov [5].

REDUCTION TO THE RIEMANN BOUNDARY VALUE PROBLEM

The functions u and v can be presented through the boundary values of piecewise holomorphic functions A(z) and B(z) on the open contour (-l,l). In accordance with the Sokhotski-Plemelj formulae one has

$$u=A^+-A^-,\ \mathbf{I}(u)=i\left(A^++A^-\right),\ v'=B^+-B^-,\ \mathbf{I}(v')=i\left(B^++B^-\right),\ u'=A'^+-A'^-,\ \mathbf{I}(u')=i\left(A'^++A'^-\right) \tag{22}$$

By substituting Eqns 22 into Eqns 18 one has a system for the determination of holomorphic functions A(z) and B(z) by their boundary values on the interval (-l,l). The equations of this system read

$$A^+(x)=G(x)A^-(x),\quad G(x)=-e^{2i\alpha(x)},\quad |x|<l \tag{23}$$

$$iB^+(x)-2A'^+(x)-\alpha'_y(x)A^+(x)=-e^{2i\alpha(x)}\left[iB^-(x)+2A'^-(x)+\alpha'_y(x)A^-(x)\right],\quad |x|<l \tag{24}$$

Eqn 24 can be reduced to the form of Eqn 23 by introducing a piecewise holomorphic functions $\beta^{\pm}(z)$ given by the Cauchy-type integral with real density α'_y

$$\alpha'_y(x)=\beta^+(x)-\beta^-(x),\quad |x|<l \tag{25}$$

Taking into account that due to Eqn 23 the following is valid on $|x|<l$

$$\alpha'_y\left(A^+-e^{2i\alpha}A^-\right)=2A^+\beta^++2e^{2i\alpha}\beta^-C^- \tag{26}$$

the following boundary values of piecewise holomorphic functions $C(z)$ can be introduced

$$C^+(x)=iB^+(x)-2A'^+(x)-2\beta^+(x)A^+(x)\quad C^-(x)=iB^-(x)+2A'^-(x)+2\beta^-(x)A^-(x),\quad |x|<l \tag{27}$$

This leads to Eqn 23 with respect to $C^{\pm}(x)$. Thus, the functions $A(z)$ and $C(z)$ can be found from the same equation; their boundary values can be determined and finally the boundary values of the function $B(z)$ are to be found by Eqn 27. Afterwards the solutions for u and v' can be obtained by Eqn 22.

The problem considered refers to the Riemann boundary value problem for open contours. It can be reduced to the case of half-plane by putting $G(x)$=1 on $|x|>l$. Then the ends of the interval will be the points of discontinuity of $G(x)$. This becomes obvious if the asymptotic behaviour of D at the crack tips is considered. Independent of load it can be written in the form

$$\sqrt{2\pi r}D=\left(K_I^{\pm}+3iK_{II}^{\pm}\right)e^{-i\theta/2}-\left(K_I^{\pm}-iK_{II}^{\pm}\right)e^{-5i\theta/2} \tag{28}$$

where K_I and K_{II} are stress intensity factors, the indices "±" refer to the right and left crack tips correspondingly and angle θ is the polar angle in local coordinate system with the origin at the crack tip. Now the argument $\alpha=\alpha(x)$ can be calculated. In particular, for points lying near the tips of the crack, the argument of D does not depend on K_I and can be determined as follows

$$\alpha(\pm l \mp 0) = \arg D(r,\pi) = \arg\left(K_{II}^{\pm}\right) = \frac{\pi}{2}\left(1-\operatorname{sgn}\left(K_{II}^{\pm}\right)\right) \quad \Rightarrow \quad e^{2i\alpha(\pm l \mp 0)} = 1 \qquad (29)$$

It is also can be seen that the argument α would gain the increment of π/2 if the point passed the crack end.

Thus the coefficient of the Riemann problem $G(x)$ for infinite contour $(-\infty,\infty)$ has discontinuities at points $x=\pm l$ and satisfies the Hölder condition everywhere except these points. However the function $\ln G$ is entered into the solution of the Riemann problem and this function determined as $i(\pi+2\alpha)$ on $(-l,l)$ may have discontinuities at points $c_k\in(-l,l)$ if the argument $\alpha=\alpha(x)$ is chosen to be $-\pi<\alpha\leq\pi$. This will lead to necessity to consider the Riemann problem which coefficient is discontinuous in more than two points. It can be shown from the elementary analysis of the function $\alpha=arg(-u+i\mathbf{I}u)$ that such discontinuous can be at the points c_k where $u\geq0$ and $\mathbf{I}u$ changes its sign when passing these points. Thus at these points the argument gains the increment $\pm2\pi$ if $u=0$ and $\pm\pi$ if $u=0$. Due to periodicity of $e^{2i\alpha}$ these jumps do not violate the Hölder condition but change the index of the homogeneous Riemann problem and hence affect the number of its solutions. The index can be calculated by the analysis of all jumps as it was done in [5]. However it is convenient to eliminate these jumps by introducing a continuos argument $\alpha=\alpha(x)$ on the Riemann surface. In this case the index of the problem can be defined by a simple formulae

$$\text{Index} = \mathrm{N}+1, \quad \mathrm{N} = \frac{1}{2\pi}\arg\left[\ln G(t)\right]\Big|_{-l}^{+l} = \frac{1}{\pi}\int_{-l}^{l}\alpha'(t)\,dt \qquad (30)$$

A similar expression for the index in the case of an arbitrary closed contour [1] has been interpreted as the doubled number of rotation of the principal stresses when traversing the contour. However here the index is calculated by summing the index due to the number of rotations of the principal stresses on the upper surface of the crack and the index due to discontinuity of G at $x=\pm l$ (which adds unity).

Now a general solution of the homogeneous Riemann problem (Eqn 23) can be derived on the basis of the solution for the infinite contour $(-\infty,\infty)$, Gakhov [5]. For the class unbounded at the ends and vanishing at infinity it can be written in the form

$$A^{\pm}(z) = \frac{e^{\Gamma^{\pm}(z)}}{\sqrt{z^2-l^2}}\frac{P_{\mathrm{N}}(z)}{(z\pm i)^{\mathrm{N}}}, \quad \Gamma^{\pm}(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\arg G(t) - 2\mathrm{N}\arg(t-i)}{t-z}\,dt \qquad (31)$$

where $P_{\mathrm{N}}(z)$ is an arbitrary polynomial of degree N with real coefficients.

The boundary values assume the form

$$A^{\pm}(x) = \frac{e^{\pm i\arg G(x)/2}e^{J(x)}}{\sqrt{x^2-l^2}^{\,\pm}}\frac{P_{\mathrm{N}}(x)}{\left(x^2+1\right)^{\mathrm{N}/2}}, \quad J(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\arg G(t) - 2\mathrm{N}\arg(t-i)}{t-x}\,dt \qquad (32)$$

Solutions for u and $\mathbf{I}u$ can be obtained by the Sokhotski-Plemelj formulae (Eqn 22). On $|x|>l$ $\arg G=0$ and the square root is continuous, hence $A^+=A^-$ and consequently $u=0$ and $\mathbf{I}u=2iA^+$. On the crack $|x|<l$ they are

$$u(x) = \cos\alpha(x)E(x), \quad \mathbf{I}u(x) = \sin\alpha(x)E(x), \quad E(x) = e^{J(x)}\left(x^2 - l^2\right)^{-1/2} P_{\mathrm{N}}(x)\left(x^2+1\right)^{-\mathrm{N}/2} \tag{33}$$

Solutions for $C^{\pm}(x)$ have the form similar to Eqn 33, with the only difference being that another polynomial should be used. Hence the corresponding expressions for the combinations in the square brackets in Eqn 18 become known. Thus the solution is completed. It should be noted that the complete solution depends upon 2N+2 real arbitrary constants. They should be determined from additional information on stress or deformation measurement.

The SIFs can be found by passing to the limit in the solutions for u and v' by the following formulae, eg. [4]

$$K_I^{\pm} - iK_{II}^{\pm} = \mp \lim_{x \to \mp l} \sqrt{2\pi|x \mp l|}\, g(x), \quad g(x) = iu(x) + \int v'(x)dx \tag{34}$$

The mode II SIF follows directly from Eqn 33. With the account for Eqn 31 this gives

$$K_{II}^{\pm} = \mp 2\sqrt{2\pi l}\, e^{J(\pm l)} P_{\mathrm{N}}(\pm l)\left(l^2+1\right)^{-\frac{\mathrm{N}}{2}} \tag{35}$$

Expressions for K_{I} are much complicated and not presented here.

CONCLUSIONS

The article presents the solution of the non-classical boundary value problem for a straight crack with the boundary conditions formulated in terms of principal stress orientations. This solution is non-unique; it depends upon a certain number of real constants determined by the index of the problem (Eqn 30).

It is also established that the orientations of principal stresses on the upper and lower surfaces of the crack cannot be given independently. They should satisfy Eqn 19-20, otherwise the problem is not solvable.

REFERENCES

1. Galybin, A.N. and Mukhamediev, Sh.A. (1999) Plane elastic boundary value problem posed on orientation of principal stresses. *J. Mechanics and Physics of Solids*. **47**, No 11, 2381.
2. Angelier, J. (1984) Tectonic analysis of fault slip data sets *J.Geophys. Res.*. **89**, 5835.
3. Muskhelishvili, N.I. (1953) *Some Basic Problems of the Mathematical Theory of Elasticity*. P.Noordhoff Ltd. Groningen-Holland.
4. Savruk, M.P (1981) *Two-dimensional problems of Elasticity for body with cracks*, Naukova Dumka, Kiev.
5. Gakhov, F.D. (1990) *Boundary value problems*. Dover Publications, Inc., New York.

APPENDIX. BOUNDARY VALUES OF THE STRESS FUNCTION *D*.

The function $g(t)$ determined by Eqn 12 is unbounded at the ends of the interval $(-l,l)$ hence the direct application of the Sokhotski-Plemelj formulae for the derivatives of the Cauchy integral should be justified. Since the integration by parts is needed to transform $g(t)$ to $g'(t)$, this requires the boundedness of $g(\pm l)$. To resolve the problem one can introduce the new density, $h(t)$, bounded at the ends as follows

$$h(t) = g(t) - \frac{at+bl}{l\sqrt{l^2-t^2}}, \quad a = \frac{u_+ - u_-}{2}, \quad b = \frac{u_+ + u_-}{2}, \quad u_{\pm} = \lim_{t\to\pm l}\sqrt{l^2-t^2}\, g(t) \tag{A1}$$

Let the following integrals be introduced

$$J_0(z)=\frac{1}{\pi}\int_{-l}^{l}\frac{1}{\sqrt{l^2-t^2}}\frac{1}{t-z}dt=\frac{-1}{\sqrt{z^2-l^2}},\quad J_1(z)=\frac{1}{\pi}\int_{-l}^{l}\frac{t}{\sqrt{l^2-t^2}}\frac{1}{t-z}dt=1-\frac{z}{\sqrt{z^2-l^2}} \tag{A2}$$

Their derivatives can be calculated by applying the residual theorem as follows

$$J_0'(z)=\frac{z}{\left(z^2-l^2\right)^{3/2}},\quad J_1'(z)=\frac{l^2}{\left(z^2-l^2\right)^{3/2}},\quad J_0''(z)=-\frac{2z^2+l^2}{\left(z^2-l^2\right)^{5/2}},\quad J_1''(z)=\frac{-3zl^2}{\left(z^2-l^2\right)^{5/2}} \tag{A3}$$

Then with the account for Eqn (A1)-(A3) and integrating by parts, the normal derivative of D takes the form

$$D_y'=-\frac{1}{2\pi}\int_{-l}^{l}\left[\frac{2h'(t)+\overline{h'(t)}}{t-z}+\frac{(t-\bar{z})h'(t)}{(t-z)^2}\right]dt-\frac{(3a+\bar{a})l+\left(3b+\bar{b}\right)z}{2\left(z^2-l^2\right)^{3/2}}+\frac{z-\bar{z}}{2}\frac{3zla+\left(2z^2+l^2\right)b}{\left(z^2-l^2\right)^{5/2}} \tag{A4}$$

With the use of the Sokhotski-Plemelj formulae for the first integrand and its generalisation (eg., [4]) for the second integrand one can obtain the following boundary value of

$$D_y'^{\pm}(x)=\pm\frac{3h'(x)+\overline{h'(x)}}{2i}-\frac{1}{2\pi}\int_{-l}^{l}\frac{3h'(t)+\overline{h'(t)}}{t-x}dt\mp\frac{i}{2}\frac{(3a+\bar{a})l+\left(3b+\bar{b}\right)x}{\left(l^2-x^2\right)^{3/2}} \tag{A5}$$

where the branch chosen for the complex-valued square root function is $\sqrt{z^2-l^2}=\pm i\sqrt{l^2-x^2}$ if z→$x\pm i0$.

Returning in Eqn A6 to the density $g(t)$ by Eqn A1 one obtains

$$D_y'^{\pm}(x)=\pm\frac{3g'(x)+\overline{g'(x)}}{2i}-\frac{1}{2\pi}\int_{-l}^{l}\frac{3g'(t)+\overline{g'(t)}}{t-x}dt+\frac{1}{2\pi}\int_{-l}^{l}\frac{(3a+\bar{a})l+\left(3b+\bar{b}\right)t}{\left(l^2-t^2\right)^{3/2}(t-x)}dt \tag{A6}$$

It can be proved that the last integral here vanishes. First the following integral of the Cauchy type can be evaluated by applying the residual theorem.

$$I(z)=\frac{1}{2\pi i}\int_{-l}^{l}\frac{(3a+\bar{a})l+\left(3b+\bar{b}\right)t}{\sqrt{t^2-l^2}\left(t^2-l^2\right)(t-z)}dt=\frac{(3a+\bar{a})l+\left(3b+\bar{b}\right)t}{\sqrt{z^2-l^2}\left(z^2-l^2\right)} \tag{A7}$$

From Eqn A7 it follows that $I^{+}(x)+I(x)=0$. The last integral in Eqn A7 is the sum of the boundary values presented by Eqn 8, hence it vanishes. Finally, the boundary value of the normal derivative of the function D assumes the standard form shown in the main text by Eqn 17.