MATHEMATICAL MODEL
OF SUBCRITICAL CREEP CRACK GROWTH

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ABSTRACT

The aim of the present paper is to suggest the mathematical model for prediction the process of initiation and subcritical crack growth under creep conditions. Based on modified fracture criterion the mathematical model describing the process of crack growth has been supposed. Crack growth rate and current crack length are described by Volterra's integral equations. The combined method of prediction the crack growth process is proposed. Using modified criterion the problem of crack increment and crack arrest at variable load is considered. The conditions, at which there is instant crack propagation through whole cross-section after load increment, are obtained.

INTRODUCTION

Creep fracture mechanics is one of the most important parts of modern fracture mechanics. It is a result of intensive development power-building machinery, aerospace industry and other technical branches.

Over the past two decades much effort has been made to correlate the creep crack growth rate $\dot{l}$ with parameters characterising the stress distribution near the crack tip. Some progress has been made in modelling the creep crack grow process and deriving the crack growth laws.

Historically several parameters have been employed to characterise creep crack growth rate. These include the stress intensity factor K, J-integral of elastic-plastic theory, the creep fracture mechanics parameter C* and others [1-3].

Typically, based on experimental data the crack growth rate $\dot{l}$ has been described by power-law relationships [1-3]:

$$\dot{l}(t) \sim K^{\alpha}, \quad \dot{l}(t) \sim J^{\beta}, \quad \dot{l}(t) \sim C^{*^{\gamma}}$$

In the present paper the mathematical model of crack creep growth is proposed. This analytical model permits to predict the process of incubation, initiation and subcritical crack growth, the phenomena of delayed fracture and instant fracture as well.

FRACTURE CRITERION

First of all, it is necessary to introduce the criteria governing creep crack growth. Having analysed the existing approaches to modelling of lifetime of metals under creep conditions, the modified fracture criterion have been chosen [4, 5]

$$\frac{\sigma_e}{1-\omega} = \sigma_*, \quad \sigma_e = \sqrt{3/2 s_{ij} s_{ij}} \tag{1}$$

where s_{ij} is the deviator of stress tensor, σ_e is the effective stress and σ_* is the ultimate stress.

This criterion includes into stress failure criterion $\sigma_e = \sigma_*$ the damage parameter of material ω. The damage accumulation and growth is prescribed by the Kachanov-Rabotnov kinetic equation [6, 7]:

$$\dot{\omega} = A\left(\frac{\sigma_1}{1-\omega}\right)^m, \quad \omega(0) = 0 \tag{2}$$

where A, m are the material constants and σ_1 is the maximum stress.

MATHEMATICAL MODEL OF CRACK GROWTH

The two-dimensional stress distribution near a crack tip under steady-state creep conditions is given by Hutchinson-Rice-Rosengren [8, 9] as follows:

$$\sigma_{ij}(r,\varphi,t) = \left(\frac{C^*(l(t))}{BI_n r}\right)^{\frac{1}{n+1}} \sigma_{ij}^o(n,\varphi) \tag{3}$$

where

$$C^* = \oint_\Gamma \left(\frac{n}{n+1} B\sigma_e^{n+1}\cos\varphi - \sigma_{ij}\nu_{ij}v_{i,1}\right)ds$$

is the path-independent integral of steady-state creep, r and φ are the crack tip polar co-ordinates, $\sigma_{ij}^o(n,\varphi)$ and I_n are well-known non-dimensional functions and constant [9, 10], B, n are material constants of power-law creep equation (Norton's law)

$$\dot{\varepsilon}_{ij} = \frac{3}{2} B\sigma_e^{n-1} s_{ij}$$

Let us introduce

$$p = \frac{m}{n+1}, \quad q = \frac{m+1}{n+1}, \quad t_* = \frac{1}{A(m+1)(\sigma_1^o(n,0)\sigma_*)^m}, \quad C_{cr}^* = B\sigma_*^{n+1} I_n \rho$$

where ρ is some microstructural parameter, and consider dimensionless variables:

$$\tau = \frac{t}{t_*}, \quad z = \frac{l(t) - l_0(t)}{\rho}, \quad c(z) = \frac{C^*(l(t))}{C_{cr}^*}$$

So, from fracture criterion (1), damage kinetic equation (2) and stress asymptotic field (3) the following equation for creep crack growth can be obtained [4, 5]:

$$\int_0^z \left(\frac{c(\xi)}{z - \xi + 1} \right)^p \tau'(\xi) d\xi = 1 - c^q(z) - \frac{\tau_0 c^p(0)}{(1+z)^p} \tag{4}$$

where τ_0 is the dimensionless time of initiation and start of a crack:

$$\tau_0 = \frac{1 - c^q(0)}{c^p(0)}$$

The combined method of prediction of the crack growth process based solving the equation (4) and finding the dependence of current crack length vs. time and applied load as well as the dependence of current crack growth rate vs. C* is further analysed.

The model proposed illustrates three types of crack growth for different values of microstructural parameter ρ and external load level. The types of crack growth are the following:

1) subcritical crack growth (incubation period, initiation and stable crack growth) before dynamic crack propagation;
2) delayed fracture (incubation period, initiation and dynamic crack propagation without stable crack growth);
3) instant fracture (instant start and dynamic crack growth without incubation period and stable crack growth).

Moreover, the model suggested describes and explains scale effect, which are well-known in experimental investigation of crack growth under creep conditions, as well as some effects of creep crack growth under variable loading.

VARIABLE LOAD

Using modified criterion the problem of crack increment at variable load is analysed. Consider two cases of variable load, the first one is the instant increase of load, and the second one is the instant partial unloading.

Let us consider that at the moment τ_1 when current crack length is z_1, the applied load increase from $\sigma_\infty^{(1)}$ to $\sigma_\infty^{(2)}$ and $c_2(z_1) > c_1(z_1)$. As it follows from kinetic equation (2), the damage does not vary at instant increase of load. The crack grows up on size Δ, which also can be found from Volterra's equation (4).

Assuming $\tau'(\xi) = 0$ for the crack grow up, one can obtain the equation for Δ:

$$\int_0^{z_1} \left(\frac{c_1(\xi)}{z_1 + \Delta - \xi + 1} \right)^p \tau'(\xi) d\xi = 1 - c_2^q (z_1 + \Delta) - \frac{\tau_0 c_1^p(0)}{(1 + z_1 + \Delta)^p} \tag{5}$$

Approximate estimation for magnitude of Δ can be obtained from previous equation by linearisation

$$\Delta \gtrsim \frac{c_2^q(z_1) - c_1^q(z_1)}{c_1^p(z_1)\tau'(z_1) + (c_1^q(z_1) - c_2^q(z_1))'} \tag{6}$$

From (6), the conditions at which there is instant crack propagation through whole cross-section after load increment are obtained; since at any point ahead of a crack tip the fracture criterion is valid.

$$\left(\frac{c_2(z_1)}{c_1(z_1)} \right)^q = 1 + \frac{1}{q} \frac{c_1^p(z_1)\tau'(z_1)}{c_2^q(z_1)} \frac{c_1(z_1)}{c_1'(z_1)}$$

A load changing, current crack length, current crack rate and structural parameter determine these conditions.

The expression for evaluation crack growth rate immediately after change of a load can be found from equation (4) and has the form

$$c_2^p(z)\tau'(z) = \frac{p\tau_0 c_1^p(0)}{(1+z)^{p+1}} - (c_2^q(z))' + p \int_0^{z_1} \frac{c_1^p(\xi)\tau'(\xi)}{(z - \xi + 1)^{p+1}} d\xi \tag{7}$$

where

$$z = z_1 + \Delta$$

The opposite situation is the following. At moment τ_1 when current crack length is z_1, the applied load drops instantly from $\sigma_\infty^{(1)}$ to $\sigma_\infty^{(2)}$ and $c_2(z_1) < c_1(z_1)$. It is shown that under partial unload crack stops and does not grow for some time $\Delta\tau$ until ahead of crack tip the modified fracture criterion (2) will be valid again. The equation for $\Delta\tau$ which is obtained from equation (4) has the form:

$$\int_0^{z_1} \left(\frac{c_1(\xi)}{z_1 - \xi + 1} \right)^p \tau'(\xi) d\xi + \int_{\tau_1}^{\tau_1 + \Delta\tau} c_2^p(z_1) d\tau = 1 - c_2^q(z_1) - \frac{\tau_0 c_1^p(0)}{(1 + z_1)^p} \tag{8}$$

From equations (4) and (8) it is possible to find analytical expression for $\Delta\tau$

$$\Delta\tau = \frac{c_1^q(z_1) - c_2^q(z_1)}{c_2^p(z_1)} \tag{9}$$

Several parameters such as $\sigma_\infty^{(2)} / \sigma_\infty^{(1)}$, z_1 and $\sigma_\infty^{(1)} / \sigma_*$ influence on $\Delta\tau$. After period of time $\Delta\tau$ crack starts to grow again and equation describing the further process of creep crack growth has the form

$$\int_0^z \left(\frac{c(\xi)}{z-\xi+1} \right)^p \tau'(\xi) d\xi = 1 - c_2^q(z) - \frac{\tau_0 c_1^p(0)}{(1+z)^p} - \frac{\Delta\tau c_2^p(z_1)}{(1+z-z_1)^p} \qquad (10)$$

It is clear that slow-down crack growth phenomena affects only at initial moment after unload for further grow and does not influence on crack growth rate when crack has grown up enough (z>>1).

NUMERICAL CALCULATION

To predict the process of slow subcritical crack growth the computer program is created. When developing the program given it was taken into account that for predicting of subcritical crack growth two approaches are traditionally used. The first one is based on the experimental data of subcritical crack growth and mapping them on series of curves -- empirical relations between crack growth rate and loading parameters K, J or C*. The second approach uses the complete finite element calculation of pre-cracked structure. The approach suggested is based on modelling of fracture process near crack tip, obtaining Volterra's integral equation for crack growth rate value and numerical solution of the equation using, if necessary, the program realising finite element method for calculating the loading parameters included into integral equation, such as stress intensity factor K, path-independent J-integral or C*-integral.

Thus, knowing a value of this parameter, it is possible to calculate magnitude of a crack increment with the help of the value of the crack rate increment using the integral solution. After that, the new configuration of a crack is defined and using the program of finite element method the new value of loading parameter is calculated, and so on. The iterative process obtained, from our point of view, uses positive legs of both approaches -- empirical and numerical. In particular, with numerical solution of integral equation using finite element method it is not necessary to calculate the stress distribution in the whole structure (like in finite element calculation of crack growth), it is enough to calculate only value of stress factor K (or J) characterising the stress near crack tip. The empirical data in the approach suggested are used on the level of strength and structural characteristics of material, used in mathematical models of crack growth afterwards. Thus, they are independent on the type of a specimen with a crack, contrary to the common empirical approach defining the data for definite specimens with cracks.

CONCLUSION

Based on modified criterion the mathematical model describing the process of crack growth is offered. Crack growth rate and current crack length are described by Volterra's integral equations. The combined method of prediction of the crack growth process is offered.

The principal feature of model presented is that for different values of microstructural parameters and external load level the analytical model describes three types of crack growth - subcritical crack growth, delayed fracture and instant fracture. The model suggested describes and explains scale effects and fact of existence of well-known effect of delayed fracture as well.

Using modified criterion the problem of crack increment at variable load is considered. The conditions, at which there is instant crack propagation through whole cross-section after load increment, are obtained.

For facilitation of numerical calculating and to predict the process of slow subcritical crack growth the computer program is created.

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