LIMIT EQUILIBRIUM OF A CURVILINEAR CRACK WITH THE PARTIAL FRICTIONAL CONTACT OF ITS SURFACES

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ABSTRACT

The 2D elasticity problem on a curvilinear crack with the partial frictional contact of its surfaces was solved by the developed numerical method using the singular integral equations of the problem. The crack limit equilibrium was analyzed taking into account formation of zones of the crack surfaces opening and closure, slip and stick. The boundaries of these zones were searched for during the problem solving. An influence of the loading trajectory in case of bi-axial loading was illustrated.

INTRODUCTION

Deformation and fracture processes of structural elements and natural objects with cracks are often accompanied by an interaction of the crack surfaces with friction. The crack surfaces contact can be caused by the features of the external loading, crack geometry and combination of material properties (e.g., in case of interface cracks). For instance, contact zones occur near the ends of a circular are crack in an elastic plane under the action of uniaxial tension in the direction of the arc axis of symmetry if the arc angle is about 215^0.

The contact of the crack surfaces leads to the redistribution of the stress and displacement fields near the crack and can change the conditions of its limit equilibrium and growth regimes. The loading history is important in case of the crack surfaces interaction especially in case of their friction contact. From the mathematical point of view the problem complexity and nonlinearity is related to the necessity to examine formation and evolution of zones of the crack surfaces opening and contact, slip and stick having unknown boundaries.

Note, that the 3D problems on the cracks with contact zones of their surfaces were analyzed in [1-3]. The 2D problems on the rectilinear and arc-wise cracks with frictional contact were considered in [4-6] and [7,8] respectively. The limit cases of a smooth contact and full stick along an arbitrary curvilinear crack were studied in [9].

MODELLING OF A CURVILINEAR CRACK WITH THE PARTIAL FRICTIONAL CONTACT OF ITS SURFACES

Statement of the boundary value problem

Let us consider an isotropic elastic plane with a curvilinear crack L. Assume that the crack surfaces can interact with friction such that the crack opening occurs at a part L_0 of the crack, while part of the crack surfaces contact $L_c=L\backslash L_0$ consists of the zones $L_{s\lambda}$ and L_{st} of their slip and stick, respectively. The boundary conditions at the crack L and additional conditions in the contact zone are the following

$$N^+ + iT^+ = N^- + iT^-, \quad (u^+ - u^-) + i(v^+ - v^-) = 0, \quad t \in L_{st}; \tag{1}$$

$$v_n^+ - v_n^- = 0, \quad T^\pm = \rho \left| N^\pm \right|, \quad t \in L_{s\lambda}; \tag{2}$$

$$N^{\pm} + i\,T^{\pm} = 0\,, \quad t \in L_0\,; \tag{3}$$

$$N^{\pm} < 0\,, \quad t \in L_c\,, \quad \mu\left|N^{\pm}\right| > \left|T^{\pm}\right|, \quad t \in L_{st} \tag{4}$$

where t is the parameter along the crack, values related to the left and right crack surfaces relative to the direction of t increasing marked by the signs ± respectively, $N^{\pm}$ and $T^{\pm}$ are the normal and shear stresses prescribed at the crack surfaces, $u^{\pm}$ and $v^{\pm}$ are the components of the crack surfaces displacements relative to the Cartesian coordinate system, $\rho = \mu\,(v_\tau^+ - v_\tau^-)/\left|v_\tau^+ - v_\tau^-\right|$, $v_n^{\pm}$ and $v_\tau^{\pm}$ are the normal and shear displacement components, μ is the Coulomb friction coefficient, $\sqrt{i} = -1$.

Basic equations

The derivative of the jumps of displacement components along the crack can be represented through two functions $g'(t) = \varphi_n' + \varphi_\tau'(t)$ where $\varphi_n(t) = 2G(1+\kappa)^{-1}[v_n]t_s'$, $\varphi_\tau(t) = -2iG(1+\kappa)^{-1}[v_\tau]t_s'$, $[v_n] = v_n^+ - v_n^-$, $[v_\tau] = v_\tau^+ - v_\tau^-$, $t_s' = dt/ds$, s is the are length; κ=3-4ν in plane strain and κ=(3-ν)/(1+ν) in plane stress; ν, G are the Poisson ratio and shear modulus of the material.

Denote by $\Delta_r = (c_r, f_r)$, r=1,…, N_{op}, $L_0 = \cup(\Delta_r)$ and by $\Delta_d = (a_d, b_d)$, d = 1,2,…, N_{sh}, $L_0 \cup L_{s\lambda} = \cup(\Delta_d)$ the intervals where the opening and mutual shear of the crack surfaces occur, respectively.

Further, the displacements and stresses have no jumps along the stick zone. Then by incorporating boundary conditions given by Eqns (2), (3) and integral representations of the complex potentials (see, e.g. [10]) we obtain the system of (N_{op}+N_{sh}) singular integral equations relative to the functions $\varphi_n'(t), t \in \Delta_r$ and $\varphi_\tau'(t), t \in \Delta_d$

$$\Omega(t') = 0\,, \quad t' \in L_f\,; \quad \operatorname{Im}\Omega(t') = \rho\left|\operatorname{Re}\Omega(t')\right|, \quad t' \in L_c\,. \tag{5}$$

$$\begin{aligned}\Omega(t') = {} & \frac{1}{2\pi}\int_{L_f}\left[\frac{2\varphi_n'(t)}{t - t'}dt + k_1(t,t')\varphi_n'(t)dt + k_2(t,t')\overline{\varphi_n'(t)dt}\right] + \\ & + \frac{1}{2\pi}\int_{L_f \cup L_c}\left[\frac{2\varphi_\tau'(t)}{t - t'}dt + k_1(t,t')\varphi_\tau'(t)dt + k_2(t,t')\overline{\varphi_\tau'(t)dt}\right] - p_0(t')\end{aligned} \tag{6}$$

where $p_0(t) = -N_0 - iT_0$, N_0 and T_0 are the stresses caused by the external loads in the continuous plane at the crack line, the regular kernels are equal to

$$k_1(t,t') = \frac{d}{dt'}\ln\frac{t - t'}{\bar{t} - \bar{t}'}\,, \quad k_2(t,t') = -\frac{d}{dt'}\frac{t - t'}{\bar{t} - \bar{t}'}\,. \tag{7}$$

The system of integral equations (Eqns (5)) has a unique solution in the class of functions unbounded in the ends of the integration interval if the conditions of the displacement components single-valuedness at the circuit of the contours Δ_r and Δ_d, respectively are satisfied

$$\int_{(c_r,f_r)}\varphi_n'(t)dt = 0\,, \quad r = 1, 2\ldots, N_{op}; \qquad \int_{(a_d,b_d)}\varphi_\tau'(t)dt = 0, \quad d = 1, 2\ldots, N_{sh} \tag{8}$$

and the equalities following from the definition of the functions $\varphi_n(t)$, $\varphi_\tau(t)$ because of the displacement reality

$$\operatorname{Im}\left[\varphi_n(t)(d\bar{t}/ds)\right] = 0\,, \quad \operatorname{Re}\left[\varphi_\tau(t)(d\bar{t}/ds)\right] = 0\,. \tag{9}$$

The system Eqns (5) can be rewritten as follows

$$\operatorname{Re}\Omega(t')=0, \quad t'\in L_0; \tag{10}$$
$$\operatorname{Im}\Omega(t')=\rho\left|\operatorname{Re}\Omega(t')\right|, \quad t'\in L_{s\lambda}\cup L_0. \tag{11}$$

Algorithm of searching for unknown boundaries

An iterative algorithm was developed for sequential searching for the zones of the crack surface opening and contact, their slip and stick as well as the normal and shear components of the crack surface displacements. The algorithm is based on the property of the displacement jumps continuity on the crack at small variations of the loads and the boundaries of the zones of the crack opening and relative shear of the crack surfaces [3]. As a result small variations of these boundaries lead to small variations of the displacement field.

The initial approximation of the zone boundaries $(a_d^{(0)}, b_d^{(0)})$, $(c_r^{(0)}, f_r^{(0)})$ is constructed according to conditions given by Eqns (4) at the loads $N = N_0$, $T = T_0$. Further, the stress intensity factors (SIF) at the ends of the crack opening zones and zones of the relative shear of the crack surfaces are calculated on the basis of the solution of the problem given by Eqns (10), (11) with additional conditions on the displacement (Eqns (8), (9)) [9]

$$k_1^{\pm}=\mu\lim_{t\to l_r^{\pm}}[\sqrt{2\left|t-l_r^{\pm}\right|}\ \operatorname{Re}g'(t)], \quad k_2^{\pm}=\pm\lim_{t\to l_d^{\pm}}[\sqrt{2\left|t-l_d^{\pm}\right|}\ \operatorname{Im}g'(t)] \tag{12}$$

where the signs ± correspond to the right and left edges of the appropriate zones along the direction of increasing the parameter t. Generally $k_1^{\pm(0)}\neq 0$ and $\left|k_2^{\pm(0)}\right|>0$.

Note, that the zones of crack opening and relative shear displacement of its surfaces can not have common end points with the exception of the case when the opening zone borders on one of the crack ends. Indeed, the opening zone is always contained in the zone of relative shear displacement of the crack surfaces. However, the relative shear zone can exist without an inner opening zone [1,3]. Taking in mind this property one can show that the common points of the opening and stick zones can not occur along the smooth curvilinear crack. Further, the sign of the normal stresses is determined by the main term of their asymptotics near the edges of the opening zone [7,8]. If $k_1^{\pm(0)}>0$, then the appropriate edge of the opening zone is shifted on a small value δ_1 to the adjacent contact zone, and it is shifted in the opposite direction, when $k_1^{\pm(0)}<0$. As a result of using this algorithm we obtain the solution nonsingular at the ends of the opening zone. Similarly singular shear stresses near the edges of the slip zone will always lead to the relative shear of the crack surfaces [7,8]. Hence, the edges of the slip zone should be shifted on a small value δ_2 in the slip direction according to the property of continuous variation of the shear displacement jump at a small variation of the slip zone boundaries [3]. The inequality $\left|k_2^{\pm(i+1)}\right|<\left|k_2^{\pm(i)}\right|$ (where i is the iteration member) needs to be fulfilled for the adequate variation of these boundaries. Thus, the problem given Eqns (8), (9), (10), (11) is solved on the i-th iteration. Then the values of the SIFs are calculated at the edges of the zones $(a_d^{(i)}, b_d^{(i)})$, $(c_r^{(i)}, f_r^{(i)})$ and the new locations of these boundaries are searched for $(a_d^{(i+1)}, b_d^{(i+1)})$, $(c_r^{(i+1)}, f_r^{(i+1)})$. The process is finished when the SIFs at the boundaries of the opening and slip zones (which do not coincide with the crack edges) become to be equal to zero within the given accuracy.

Numerical method

The method of mechanical quadratures based on the polynomial interpolation formulae and quadrature formulae for singular integrals [1-3] was used for the numerical solving of the system of singular integral equations.

Taking in mind that the derivative of the displacement jump has the square root singularity while the integral equations contain the Cauchy singular kernels one can use the Gauss quadrature formulae

$$\int_{-1}^{1}\frac{f(\xi)d\xi}{(\xi-\eta)\sqrt{1-\xi^2}}=\frac{\pi}{n}\sum_{k=1}^{n}\frac{f(\xi_k)}{\xi_k-\eta}+\pi f(\eta)\frac{U_{n-1}(\eta)}{T_n(\eta)}, \quad \int_{-1}^{1}\frac{f(\xi)d\xi}{\sqrt{1-\xi^2}}=\frac{\pi}{n}\sum_{k=1}^{n}f(\xi_k) \tag{13}$$

where $U_{n-1}(\eta) = \sin(n \arccos \eta) \big/ \sqrt{1-\eta^2}$ is the Chebyshev polynomial of the second kind and the values ξ_k are equal to the roots of the Chebyshev polynomials of the first kind $T_n(\xi) = \cos(n \arccos \xi)$ such that $\xi_k = \cos[\pi(2k-1)/2n]$ (k = 1, 2, …, n). As a result the system of the integral equations is reduced to the system of algebraic ones relative to the values of unknown functions in the discrete set of points ξ_k. The Lagrange interpolational polynomial

$$f(\xi) = \frac{1}{n} \sum_{k=1}^{n} (-1)^{k+1} f(\xi_k)\, T_n(\xi) \sqrt{1-\xi_k^2} \big/ (\xi - \xi_k) \tag{14}$$

was used to construct the unknown functions through their values in the points ξ_k.

The feature of the used method suggested in [4] consists in independent quantization with respect to variables ξ and η while usually the roots of the Chebyshev polynomials of the second kind are chosen as $\eta = \eta_m$ (m = 1, 2, …, n-1). In our method quantization with respect to η was performed on the sets related to the domain of definition of the functions φ_n (t) and φ_τ (t). Such representation enables us to characterize independently opening and shear of the crack surfaces in each of the zones where these relative displacements occur.

Numerical results

As the examples let us consider the results of the numerical calculations for the parabolic or elliptic arc-wise cracks in plane under the biaxial loading by the loads p and q ($p \le 0$, $q > 0$) such that the components p is inclined at the angle α to the axis x. The parabolic and elliptic arcs parameterization was written as follows $t = \omega(\xi) = \lambda\, [\xi + i\varepsilon(\xi^2 - 1)]$ and $\omega(\xi) = \lambda\, [2\xi - i\varepsilon(1 - \xi^2)]/(1 + \xi^2)$, ε characterizes the crack curvature. The crack ends are located in the points $(\pm\lambda, 0)$. The calculated dependencies of the SIF $\left(k_2 / q\sqrt{\lambda}\right)$ and length of the opening zone (S_0/S_{cr}) where S_{cr} is the crack length on the ratio (p/q) at $\alpha = 0$ are given in Figs 1, 3 and 2, 4 respectively, for the parabolic and elliptic arc-wise crack. The curves (1), (2), (3) are related to the values of the friction coefficient μ = 0; 0.2; 0.4. The SIF dependencies on the parameter ε are given for ε = 1 (curve (a)); 1.5 (b); 2 (c). For the parabolic crack (Fig. 2) the length of the opening zone monotonically decreases with growth of the friction coefficient for all ε. The length of each of the contact zones equals $S_c = (S_{cr} - S_0)/2$. The relative length of the zone of the mutual shear displacements of the crack surfaces (S_{sh}/S_{cr}) = 0.958 at μ = 0.4, ε = 2, |p/q| = 8.

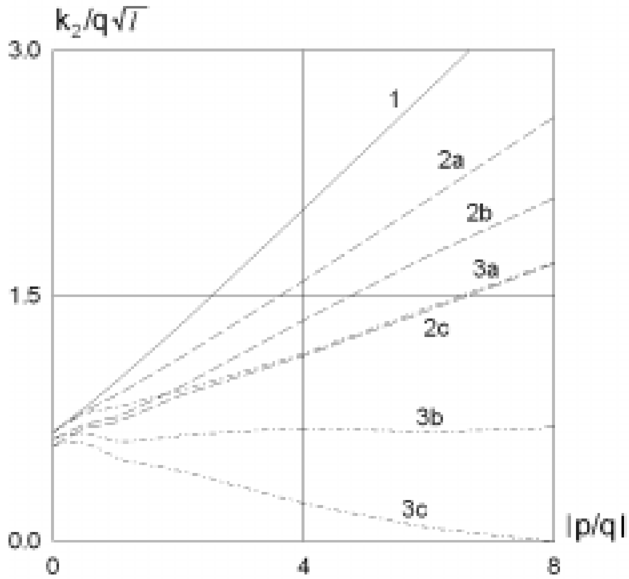

Figure 1: SIF of the shear stress for the parabolic arc-wise crack

Comparing Figs 1 and 3 one can conclude that the orientation of the crack ends essentially changes the form of the SIF dependencies on the applied loads (e.g., the function k_2 (p/q) becomes strongly nonlinear).

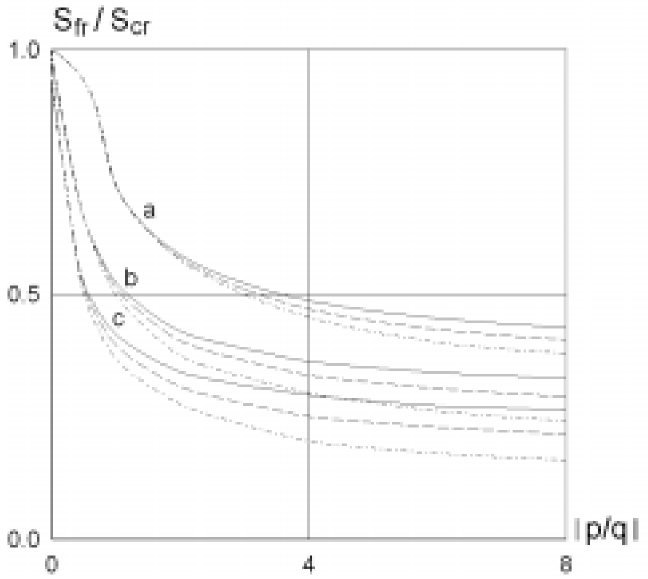

Figure 2: Relative length of the opening zone for the parabolic arc-wise crack

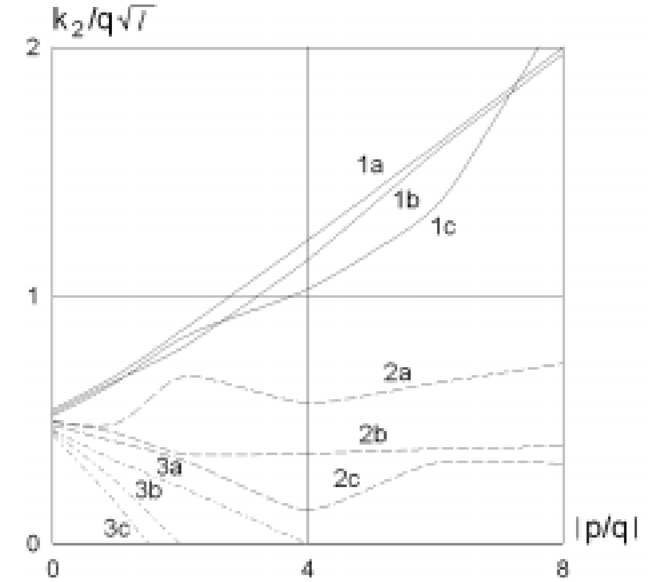

Figure 3: SIF of the shear stress for the elliptic arc-wise crack

Further, the stick zones are formed near the ends of elliptic arc-wise crack even at $|p/q| > 4$ ($\varepsilon = 1$). Opposite to the case of the parabolic crack, small contact zones occur near the ends of the elliptic one at p=0. The dependence of the relative length of the relative shear of the elliptic arc crack surfaces (S_{sh}/S_{cr}) on the loading parameters $|p/q|$ is given in Fig. 5. It was aforementioned that the stick zones of size $S_{st}=(S_{cr} - S_{sh})/2$ occur near the crack ends. The length of the stick zone increases with growth of the parameter $|p/q|$.

If the solution of the boundary value problem and unknown opening, slip and stick zones were searched for, one can analyze the crack limit equilibrium using known criteria. For instance, in the aforeconsidered examples of cracks and bi-axial loading only the SIF $k_2 \neq 0$. Hence, one can assume that in these cases the limit equilibrium of the cracks can be evaluated according to the criterion $k_2 = k_{2c}$, where k_{2c} is the fracture toughness relative to the transverse shear.

Close to linear monotone k_2 increasing with increasing the loading parameter $|p/q|$ is observed for the parabolic arc-wise crack at small values of the curvature and friction coefficient (Fig. 1). The opposite effect of the SIF k_2 decreasing with $|p/q|$ increasing occurs at the increasing of the curvature and friction coefficient since the crack surfaces slipping is decelerated by growing friction. Hence, in this case the growth of the loading parameter will not lead to attaining the crack limit equilibrium.

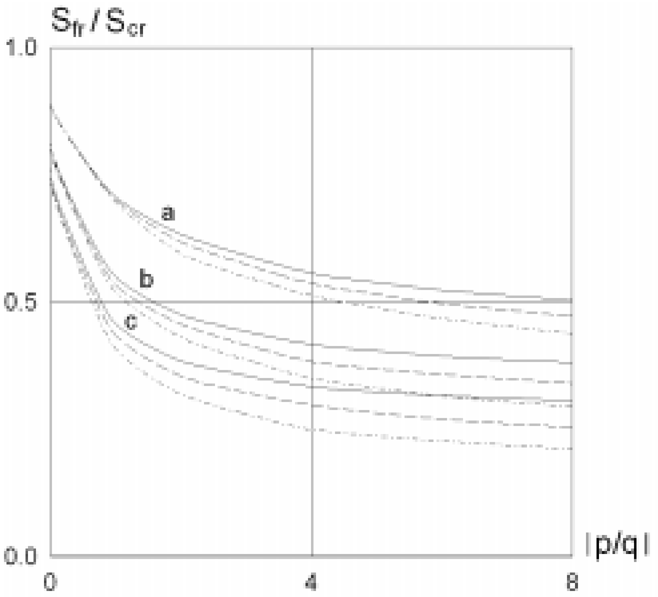

Figure 4: Relative length of the opening zone for the elliptic arc-wise crack

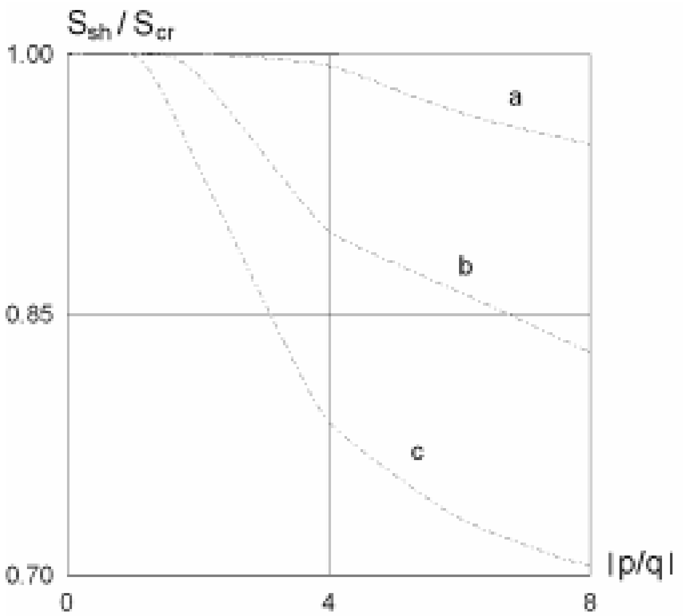

Figure 5: Relative length of the zone of relative shear displacements for the elliptic arc-wise crack

Monotone k_2 increasing and decreasing with |p/q| increasing are observed for the elliptic arc-wise crack in absence of friction (and $\forall\varepsilon$) and at the friction coefficient $\mu = 0.4$, respectively (Fig. 3). On the other hand, nonmonotone dependence k_2 (|p/q|) is appropriate to the intermediate values of the friction coefficient, e.g. at $\mu = 0.2$.

The results of calculations demonstrate the complex nature of the friction and shear loading influence on the possible regimes of attaining the crack limit equilibrium.

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