FATIGUE FAILURE ASSESSMENT UNDER MULTIAXIAL LOADING

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ABSTRACT

According to the so-called critical plane approach, the plane where the fatigue failure assessment should be performed is determined by maximising the amplitudes and/or values of some stress components. In the present paper, the critical plane orientation is proposed to be correlated with the averaged principal stress directions deduced through the weight function method. Then the fatigue failure assessment is performed by considering a function of the stress components acting on the critical plane. The results derived by applying this criterion or some other critical plane criteria commonly used are compared with experimental data related to different brittle (hard) metals under in-phase or out-of-phase sinusoidal biaxial normal and shear stress states.

INTRODUCTION

Several criteria developed during the last decades to predict whether fatigue failure under multiaxial loading may occur or not are generally aimed at reducing a given multiaxial stress state to an equivalent uniaxial stress condition (e.g. see review in Ref. [1]). Some of these criteria are based on the so-called critical plane approach, according to which the fatigue failure assessment is performed in a plane where the amplitude or the value of some stress components or a combination of them attains its maximum [2-5]. Alternatively, the position of the critical plane may be correlated with that of the principal stress directions, but, since such directions under fatigue loading are generally time-varying, averaged principal stress directions should be considered [6-8].

In the following, a new criterion is proposed which correlates the critical plane orientation with the mean principal stress directions determined through the weight function method. Then the fatigue failure assessment is performed by considering a quadratic function of the shear amplitude and the maximum normal stress acting on the critical plane. Finally, such a criterion is applied to some experimental tests on brittle (hard) metals under in-phase or out-of-phase sinusoidal biaxial normal and shear stress states. For these materials, the ratio between the endurance limit under fully reversed torsion and that under fully reversed bending falls within the following range : $1 \big/ \sqrt{3} \ \le \ \tau_{af} \big/ \sigma_{af} \ \le \ 1$.

FATIGUE CRITERIA BASED ON THE CRITICAL PLANE APPROACH

Let us consider the plane stress condition of biaxial normal and shear stresses at the generic point P of a cylindrical body (Figure 1a) subjected to synchronous out-of-phase sinusoidal loading :

$$\begin{aligned} \sigma_{xx} &= \sigma_t(t) = \sigma_{t,a} \sin(\omega t - \alpha) + \sigma_{t,m} \\ \sigma_{yy} &= \sigma_l(t) = \sigma_{l,a} \sin(\omega t) + \sigma_{l,m} \\ \sigma_{xy} &= \tau(t) = \tau_a \sin(\omega t - \beta) + \tau_m \end{aligned} \tag{1}$$

where the subscripts *t*, *l*, *a* and *m* stand for tangential (circumferential), longitudinal, amplitude and mean value, respectively, while the other components of the stress tensor are equal to zero.

An elementary material plane Δ, passing through point P, and two orthogonal unit vectors, **u** and **v**, on this plane are considered (Figure 1b). Let us assume the direction of the vector **u** as the intersection between Δ and the plane defined by the normal vector **w** and the Z-axis, so that Puvw forms a right-hand orthogonal coordinate system. The direction cosines of the w-axis can be computed, with respect to the PXYZ frame, as a function of two angles, φ and ϑ, in a spherical coordinate system ($0° \leq \varphi < 360°$; $0° \leq \vartheta \leq 180°$) : $l_w = \sin\vartheta \cos\varphi$, $m_w = \sin\vartheta \sin\varphi$, $n_w = \cos\vartheta$ (Figure 1b). Furthermore, the direction cosines of the u- and v-axis are equal to $l_u = \cos\vartheta \cos\varphi$, $m_u = \cos\vartheta \sin\varphi$, $n_u = -\sin\vartheta$ and $l_v = -\sin\varphi$, $m_v = \cos\varphi$, $n_v = 0$, respectively.

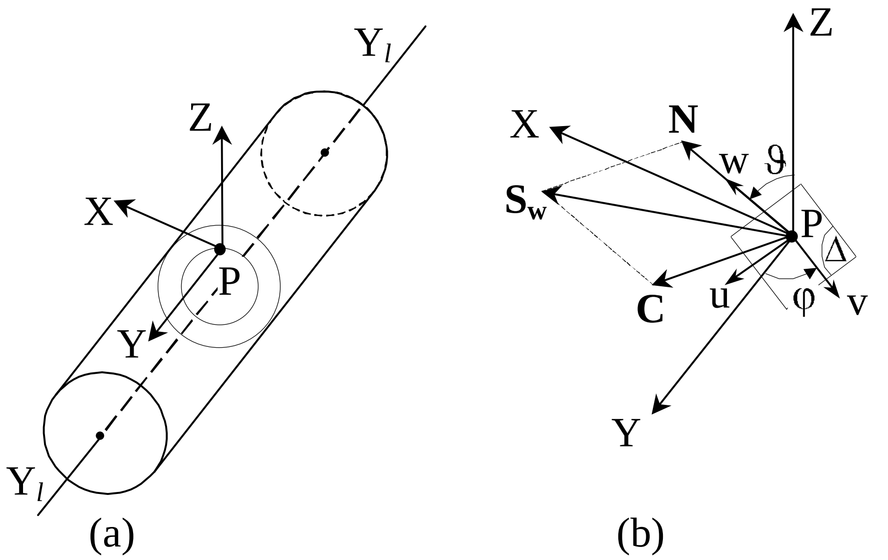

Figure 1: (a) PXYZ coordinate system; (b) Puvw coordinate system, with the w-axis normal to the plane Δ

The stress vector $\mathbf{S}_w$ acting at point P of the plane Δ (Figure 1b) can be computed if the stress tensor $\boldsymbol{\sigma}$ is known:

$$\mathbf{S_w} = \boldsymbol{\sigma} \cdot \mathbf{w} \Rightarrow [S_w] = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & 0 \\ \sigma_{xy} & \sigma_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} l_w \\ m_w \\ n_w \end{bmatrix} \Rightarrow [S_w] = \begin{bmatrix} S_{w,x} \\ S_{w,y} \\ S_{w,z} \end{bmatrix} = \begin{bmatrix} \sin\vartheta\,(\sigma_t \cos\varphi + \tau \sin\varphi) \\ \sin\vartheta\,(\sigma_l \sin\varphi + \tau \cos\varphi) \\ 0 \end{bmatrix} \tag{2}$$

and the scalar value $N(t)$ of the normal stress is given by

$$N(t) = \mathbf{w} \cdot \mathbf{S_w} \Rightarrow N(t) = l_w S_{w,x} + m_w S_{w,y} + n_w S_{w,z} = \sin^2\vartheta \left[\sigma_t \cos^2\varphi + \sigma_l \sin^2\varphi + \tau \sin 2\varphi\right] \tag{3}$$

The mean value N_m and the amplitude N_a of $N(t)$ can be determined by substituting the stress components (Eqns 1) into Eqn 3 [8].

By recalling Eqn 2, the shear stress vector **C** lying on the plane Δ is computed :

$$\mathbf{C} = \mathbf{S_w} - \mathbf{N} = \mathbf{S_w} - (\mathbf{w} \cdot \mathbf{S_w}) \cdot \mathbf{w} \Rightarrow \quad [C] = \begin{bmatrix} C_x \\ C_y \\ C_z \end{bmatrix} =$$

$$\begin{bmatrix} \sin\vartheta \left[\cos\varphi \left(\sigma_t \cos^2\vartheta + (\sigma_t - \sigma_l) \sin^2\vartheta \sin^2\varphi \right) + \tau \sin\varphi \left(1 - 2\sin^2\vartheta \cos^2\varphi \right) \right] \\ \sin\vartheta \left[\sin\varphi \left(\sigma_l \cos^2\vartheta + (\sigma_l - \sigma_t) \sin^2\vartheta \cos^2\varphi \right) + \tau \cos\varphi \left(1 - 2\sin^2\vartheta \sin^2\varphi \right) \right] \\ -\sin^2\vartheta \cos\vartheta \left(\sigma_t \cos^2\varphi + \sigma_l \sin^2\varphi + \tau \sin 2\varphi \right) \end{bmatrix} \tag{4}$$

The direction of the shear stress vector **C** is generally time-varying and, therefore, the definition of the amplitude and mean value of this vector is a complex problem. Papadopoulos [9] has recently proposed to examine the components of **C** along the u- and v-axis. By recalling Eqns 1 and 4, such components can be written in the following form :

$$\begin{aligned} C_u &= f \sin(\omega t) + g \cos(\omega t) + C_{u,m} \\ C_v &= p \sin(\omega t) + q \cos(\omega t) + C_{v,m} \end{aligned} \tag{5}$$

where $C_{u,m}$ and $C_{v,m}$ are the mean values of the components of **C** along the u- and v-axis, whereas the functions f, g, p and q depend on the phase angles, α and β, and the orientation angles, φ and ϑ, of the plane Δ [8]. Eqns 5 are the parametric equations of the ellipse described by the tip of the shear stress vector **C** on the plane Δ during a loading cycle. This ellipse is centred at point ($C_{u,m}$;$C_{v,m}$), and its semi-axes are funcions of f, g, p and q. The mean value C_m of the shear stress **C** is equal to $\sqrt{C_{u,m}^2 + C_{v,m}^2}$, whereas the amplitude of **C** coincides with the major semi-axis C_a of such an ellipse.

The analytical stress components determined above can be used to apply common fatigue failure criteria based on the critical plane approach to the case of a sinusoidal biaxial normal and shear stress state. Accordingly, the fatigue failure assessment is performed in two steps. Firstly, the critical plane is determined by maximising the amplitudes and/or values of some stress components. For example, the criteria proposed by Findley [2], Matake [4] and McDiarmid [5] can be expressed as follows :

$$(\varphi_c, \vartheta_c): \qquad \max_{(\varphi,\vartheta)} \left\{ C_a(\varphi,\vartheta) + A\, N_{\max}(\varphi,\vartheta) \right\} \tag{6}$$

where the maximum value $N_{\max}$ of the normal stress $N(t)$ is equal to the sum of its mean value and amplitude, i.e. $N_{\max} = N_m + N_a$. Secondly, the fatigue failure assessment is carried out by employing some stress components acting on the critical plane deduced in the first step :

$$C_a(\varphi_c, \vartheta_c) + B\ N_{\max}(\varphi_c, \vartheta_c) \leq D \tag{7}$$

where A, B and D are material constants which present a different expression for each criterion mentioned.

A NEW CRITICAL PLANE CRITERION

Orientation of the Fatigue Fracture Plane and the Critical Plane

A theoretical procedure has recently been developed in order to determine the mean principal stress directions through the weight function method [6-8]. At a given time instant, the principal stresses, σ_n, with $n = 1,\ 2,\ 3,$

are the eigenvalues of the stress tensor at that time instant, whereas the eigenvectors represent the nine principal direction cosines l_n, m_n, n_n. Let us assume that $\sigma_1 \geq \sigma_2 \geq \sigma_3$, that is to say, the directions of maximum and minimum principal stresses are called 1-axis and 3-axis, respectively. The orthogonal coordinate system P123 with origin at point P and axes coincident with the principal stress directions can also be defined through the "principal" Euler angles, ϕ, θ, ψ ($0° \leq \phi < 360°$; $0° \leq \theta \leq 180°$; $0° \leq \psi < 360°$). Such Euler angles can be obtained from the above principal direction cosines, and their ranges at the end of a two-stage procedure proposed in Refs [6-8] are reduced as follows : $0° \leq \phi, \theta \leq 90°$ and $-90° \leq \psi \leq 90°$, in order to average correctly the results determined for different time instants.

Now examine the stress tensor consisting of the normal stresses $\sigma_t(t), \sigma_l(t)$ and the shear stress $\tau(t)$ given in Eqns 1. Since every stress component is time-varying, we can compute the reduced principal Euler angles $\phi(t), \theta(t)$ and $\psi(t)$ at each time instant t. Then the mean directions $\hat{1}$, $\hat{2}$ and $\hat{3}$ of the principal stress axes are determined by averaging the instantaneous values of the reduced principal Euler angles through appropriate weight functions, $W(t_k)$:

$$\hat{\phi} = \frac{1}{W} \sum_{t_1}^{t_N} \phi(t_k)\, W(t_k) \qquad \hat{\theta} = \frac{1}{W} \sum_{t_1}^{t_N} \theta(t_k)\, W(t_k)$$
$$\hat{\psi} = \frac{1}{W} \sum_{t_1}^{t_N} \psi(t_k)\, W(t_k) \qquad W = \sum_{t_1}^{t_N} W(t_k) \tag{8}$$

where W represents the summation of the weights $W(t_k)$, with t_k from t_1 to t_N . The following weight function which accounts for the effect of the maximum principal stress $\sigma_1(t_k)$ is adopted :

$$W(t_k) = \begin{cases} 0 & \text{if} \quad \sigma_1(t_k) < c\,\sigma_{af} \\ & \qquad\qquad 0 < c \leq 1 \\ \left(\dfrac{\sigma_1(t_k)}{\sigma_{af}}\right)^{m_\sigma} & \text{if} \quad \sigma_1(t_k) \geq c\,\sigma_{af} \end{cases} \tag{9}$$

the physical meaning of which has been discussed in Refs [6-8]. Note that m_σ is a coefficient which depends on the slope of the S-N curve for fully reversed bending.

Several authors have proposed methods to predict the orientation of the plane where a fatigue crack may appear (fatigue fracture plane). For instance, according to McDiarmid [10], the fracture plane under out-of-phase sinusoidal bending and torsion coincides with the plane on which the maximum principal stress achieves its greatest value with respect to time. The correlation between the experimental fatigue fracture plane and the averaged principal stress directions has been analysed for hard metals under out-of-phase sinusoidal bending and torsion [8] and random proportional bending and torsion [11]. On the basis of the test data examined, the normal to the fracture plane seems to agree with the weighted mean direction $\hat{1}$ of the maximum principal stress by employing the weight function in Eqn 9, as is shown in the following.

Then the correlation between the above weighted mean direction $\hat{1}$ and the normal w_c to the critical plane on which the fatigue failure assessment should be performed (critical plane) is discussed. The following formula is proposed :

$$\delta = 45\,\frac{3}{2}\left[1 - \left(\frac{\tau_{af}}{\sigma_{af}}\right)^2\right] \tag{10}$$

where δ is the angle, expressed in degrees, between $\hat{1}$ and w_c. This angle is equal to 0° for $\tau_{af} / \sigma_{af} = 1$ (extremely hard metals), whereas it is equal to 45° for $\tau_{af} / \sigma_{af} = 1/\sqrt{3}$ (threshold value between hard and mild metals). As a consequence of the above assumption on δ and the conclusion drawn at the end of the previous paragraph, the critical plane is close to the fatigue fracture plane for very brittle materials, while the two planes form an angle equal to about 45° for materials with the endurance limit ratio, τ_{af} / σ_{af}, tending to the brittle-ductile threshold value.

Fatigue Failure Assessment

The fatigue failure assessment can be carried out through a quadratic combination of the maximum normal stress ($N_{\max} = N_m + N_a$) and the shear stress amplitude (C_a), acting on the critical plane :

$$\left(\frac{N_{\max}}{\sigma_{af}}\right)^2 + \left(\frac{C_a}{\tau_{af}}\right)^2 \leq 1 \tag{11}$$

This inequality takes into account some established experimental findings. First of all, as was observed by Gough et al. [12], the mean shear stress C_m does not affect the fatigue life of the test specimens. Moreover, a tensile mean normal stress N_m strongly reduces the fatigue resistance of metals, while a compressive N_m has a beneficial effect. The fatigue criterion of Eqn. 11 corresponds to an equivalent stress amplitude, σ_{eq}, to be compared with the fatigue limit σ_{af} :

$$\sigma_{eq} = \sqrt{N_{\max}^2 + \left(\frac{\sigma_{af}}{\tau_{af}}\right)^2 C_a^2} \leq \sigma_{af} \tag{12}$$

APPLICATIONS

The present fatigue criterion is applied to some experimental results related to synchronous in-phase or out-of-phase sinusoidal loading for round bars under bending and torsion [13]. Nishihara and Kawamoto employed specimens of different materials, such as mild steel with 0.15% C content and grey cast iron with 3.87% C content. Some mechanical properties of the mild steel are as follows : σ_u = ultimate tensile strength = 704.1 MPa, σ_{af} = 313.9 MPa, τ_{af} = 196.2 MPa, m_σ = 8.7, while those of the grey cast iron are : σ_u = 230.0 MPa, σ_{af} = 96.1 MPa, τ_{af} = 91.2 MPa, m_σ = 19.4. Note that, on the basis of the values of the ratio τ_{af} / σ_{af}, the former material analysed is close to the the brittle-ductile threshold, while the latter is very brittle.

The phase angle β (Eqns 1) is equal to 0° (in-phase) or 60°, 90° (out-of-phase), and the mean stresses are equal to zero; different values of the amplitude ratio ($\tau_a/\sigma_{l,a}$) have been examined (Tables 1 and 2). It needs to be underlined that all these loading cases correspond to the limit state of non-fracture of the specimens for a cycle number of the order of one million.

First of all, the experimental fatigue fracture plane orientation is compared with the theoretical predictions of the present criterion. The experimental fatigue fracture plane is described by the angle η_{exp} between the normal to the cracked plane and the longitudinal axis (Y_l-axis in Figure 1a, with Y_l parallel to Y) of the specimen, while the theoretical angle, η_{cal}, between the Y_l-axis and the weighted mean direction $\hat{1}$ of the maximum principal stress, is calculated by assuming that the coefficient c into Eqn 9 is equal to 0.5 (Tables 1 and 2). The evaluation of the fracture plane orientation according to McDiarmid [10] is also reported, where η_{cal} is the angle between the Y_l-axis and the normal to the plane on which the maximum principal stress achieves its greatest value with respect to time. The predictions through the criterion of the present authors are generally satisfactory, especially in the case of low values of the phase angle β.

TABLE 1

EXPERIMENTAL AND THEORETICAL FATIGUE FRACTURE PLANE ORIENTATION, η, AND ERROR INDEX, I, FOR MILD STEEL SPECIMENS [13]

Test *No.*	$\sigma_{l.a}$ (MPa)	τ_a (MPa)	$\tau_a/\sigma_{l.a}$	β (°)	*Fracture plane* *Exp.* η_{exp} (°)	*Present* η_{cal} (°)	*McDiarmid* η_{cal} (°)	*Error Index, I* (%) *Present*	*Findley*	*Matake*	*McDiarmid*
1	194.3	0.0	0.0	0	0	0	0	-18	-17	-17	-20
2	245.3	0.0	0.0	0	0	0	0	4	4	4	1
3	235.6	48.9	0.2	0	12	12	11	7	7	6	4
4	187.3	93.6	0.5	0	22	23	23	5	8	8	5
5	101.3	122.3	1.2	0	30	34	34	-1	2	2	1
6	0.0	166.8	∞	0	8	45	45	21	21	21	21
7	0.0	142.3	∞	0	45	45	45	4	4	4	4
8	201.1	100.6	0.5	60	8	18	17	7	7	1	-2
9	194.2	97.1	0.5	60	12	18	17	3	4	-3	1
10	105.2	126.8	1.2	60	22	35	35	1	5	-1	-2
11	108.9	131.5	1.2	60	8	35	35	5	9	2	1
12	244.8	50.7	0.2	90	0	0	0	6	5	5	2
13	235.6	48.9	0.2	90	0	0	0	2	1	1	-2
14	235.8	117.9	0.5	90	8	8	0	18	13	*	-1
15	208.1	104.1	0.5	90	8	8	0	4	0	*	-4
16	112.6	136.0	1.2	90	39	39	39	9	11	-1	-1
17	116.4	140.5	1.2	90	8	39	39	13	15	2	2

* The critical plane is undetermined

TABLE 2

EXPERIMENTAL AND THEORETICAL FATIGUE FRACTURE PLANE ORIENTATION, η, AND ERROR INDEX, I, FOR GREY CAST IRON SPECIMENS [13]

Test *No.*	$\sigma_{l.a}$ (MPa)	τ_a (MPa)	$\tau_a/\sigma_{l.a}$	β (°)	*Fracture plane* *Exp.* η_{exp} (°)	*Present* η_{cal} (°)	*McDiarmid* η_{cal} (°)	*Error Index, I* (%) *Present*	*Findley*	*Matake*	*McDiarmid*
1	103.0	0.0	0.0	0	0	0	0	7	7	7	-32
2	93.2	0.0	0.0	0	0	0	0	-4	-3	-3	-39
3	95.2	19.7	0.2	0	12	12	11	3	3	1	-33
4	83.4	41.6	0.5	0	25	23	23	4	6	6	-26
5	56.3	68.0	1.2	0	34	34	34	5	8	5	-13
6	0.0	98.1	∞	0	45	45	45	2	8	8	8
7	0.0	94.2	∞	0	49	45	45	-2	3	3	3
8	93.7	46.9	0.5	60	16	18	17	12	17	4	-22
9	67.6	81.6	1.2	60	33	35	35	17	23	6	-4
10	99.6	20.6	0.2	90	0	0	0	5	8	8	-34
11	104.2	21.6	0.2	90	0	0	0	10	13	13	-31
12	97.1	48.6	0.5	90	0	8	0	14	19	*	-26
13	75.1	90.6	1.2	90	38	39	39	16	23	-1	-1
14	71.3	86.1	1.2	90	37	39	39	11	17	-6	-6

* The critical plane is undetermined

Now the critical plane criterion previously discussed and those of other authors are applied to the experimental tests reported in Ref. [13]. The present criterion (Eqn 11) suggests that, by plotting the shear stress amplitude C_a against the maximum normal stress $N_{\max}$ acting on the critical plane, fatigue failure occurs for the points with coordinates ($N_{\max}, C_a$) which lie out of the ellipse with semi-axes equal to σ_{af} and τ_{af} .

Figures 2 and 3 show a good correlation between this theoretical ellipse (continuous line) and the test results related to the failure limit state, since most of the experimental points fall very close to such an ellipse, between two dashed curves representing an error of ± 10 % (determined as is discussed below). Furthermore, nearly all the predictions are conservative because the test points lie out of the safety domain.

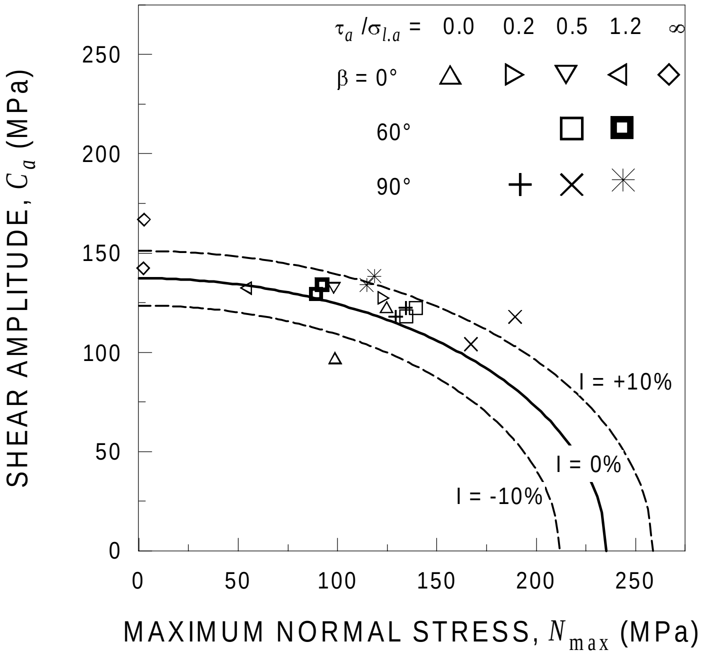

Figure 2: Shear stress amplitude vs maximum normal stress acting on the critical plane: theoretical predictions and experimental results for mild steel specimens [13]

If the left- and right-hand sides of the inequality expressing a given critical plane criterion (Eqn 7) are equal to each other for a certain test, the criterion exactly predicts the result of such a test (as mentioned before, all the experimental cases considered correspond to the non-fracture limit state). Therefore, the deviation between theoretical and experimental results can be expressed by an error index *I*, defined as the relative difference (with respect to the right hand side) between the left and right hand sides of the inequality in Eqn 7. In the case of the criterion herein proposed, the error index is equal to $(\sigma_{eq} - \sigma_{af}) / \sigma_{af}$ (see Eqn 12).

The last four columns of Tables 1 and 2 show the values of the error index for the present criterion and for those of Findley [2], Matake [4] and McDiarmid [5]. Note that a positive value of I represents a conservative prediction. It can be observed that, for all the different criteria, the error is very low, except a few cases, and independent of β.

CONCLUSIONS

A theoretical procedure has been developed to determine the averaged principal stress axes by employing the weight function method. The weighted mean direction of the maximum principal stress has been used both to predict the orientation of the fatigue fracture plane and to deduce the critical plane where the fatigue failure assessment should be performed. Then, a criterion based on the maximum normal stress and the shear stress amplitude acting on the critical plane has been presented to carry out such an assessment.

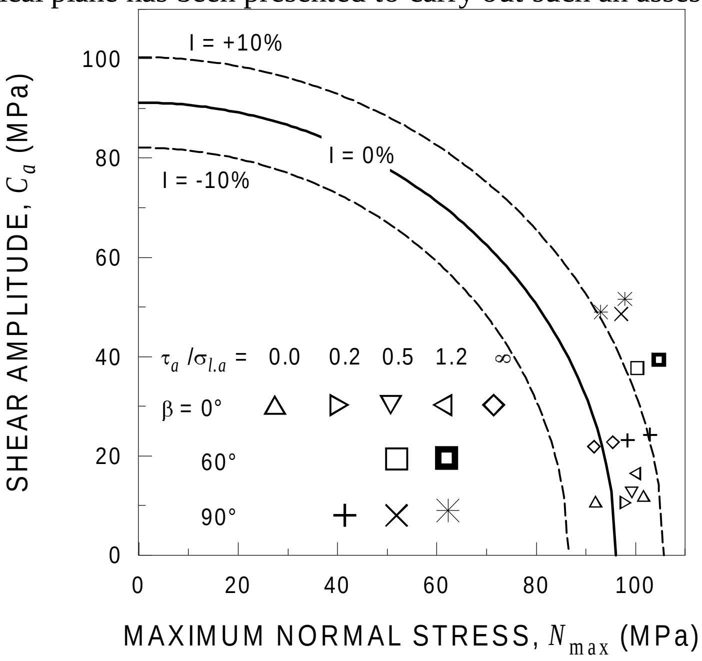

Figure 3: Shear stress amplitude vs maximum normal stress acting on the critical plane: theoretical predictions and experimental results for for grey cast iron specimens [13].

The criterion herein proposed and other critical plane criteria commonly used have been applied to some experimental tests on brittle (hard) metals under in-phase or out-of-phase sinusoidal biaxial normal and shear stress states. The normal to the experimental fracture plane agrees with the weighted mean direction of the maximum principal stress, particularly in the case of low values of the phase angle between the applied loads. Furthermore, the predictions based on the present fatigue failure assessment are generally conservative, and agree with the experimental results quite well.

Acknowledgements

The authors gratefully acknowledge the research support for this work provided by the Italian Ministry for University and Technological and Scientific Research (MURST) and the Italian National Research Council (CNR).

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