# ANISOTROPIC CYLINDRICAL SHELL WITH SURFACE CRACK

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## ABSTRACT

The procedure for solution of the problems on the stressed state and limit equilibrium of anisotropic cylindrical shell with surface cracks is suggested. The equations of refined Tymoshenko-type shell theory are taken as the input ones. The essence of method lies in that a three-dimensional problem for the shell with a surface crack of given sizes is reduced to a two-dimensional one for a shell with a through crack of unknown length using the generalised $\delta_c$ -model analogue. Here to the crack edges the unknown efforts and moments are applied and they satisfy the plasticity conditions for thin shells. On the basis of distortion method the two-dimensional problem is reduced to a system of singular integral equations with unknown limits of integration and discontinuous right-hand parts, which contain the unknown efforts and moments values. This system is complemented by the plasticity conditions, and efforts and moments boundedness conditions. The numerical algorithm for solution of this system is developed. The numerical analysis for dependence of the crack front opening on its size, physical-mechanical parameters of the shell is carried out. The relation between the critical crack sizes and loading is determined on the basis of deformable criteria used in the $\delta_c$ -model.

## INTRODUCTION

Investigation of the stressed-strained state and limit equilibrium of anisotropic shells (including the orthotropic ones) with cracks is carried out within the scope of elasticity theory. In addition, namely the Kirchhoff shell theory equations are taken as the input ones. Application of the classical theory in calculating the anisotropic-material shells does not allow to take into account inherent in them effects, connected with finite shear rigidity of thin-walled elements. Besides, the classical shell theory does not allow to satisfy completely the natural boundary conditions on the crack contour. So, we shall use the refined shell theory equations, based on the Tymoshenko hypotheses, accounting for the stated above peculiarities, to solve the problem on limit equilibrium for an anisotropic elastoplastic shell with a surface crack.

Note that construction of solution to the classical three-dimensional problem for a shell with a crack, when two systems of three-dimensional equations in two regions - elastic and plastic with unknown boundary between them, is a very complicated mathematical problem. Therefore, for the case, when plastic strains by the front of a non-through crack develop as a thin strip through the whole shell thickness, we shall use the $\delta_c$ -model analogue. This means, that the plastic strains thin strip is replaced by the surfaces of elastic generalised displacement discontinuities, and the plastic strains zone reaction on the elastic zone we shall replace by the unknown efforts and moments , that satisfy the thin shell plasticity conditions. Thus, the three-dimensional elastoplastic problem for the shell with a surface crack of given sizes is reduced to the two-dimensional one on the limit equilibrium of elastic shell with a crack of unknown length, to which edges the unknown efforts and moments satisfying the plasticity conditions are applied.

## MATHEMATIAL MODEL OF SURFACE CRACK IN A THIN SHELL

Consider a thin cylindrical shell, related to the curvature lines $\alpha,\beta$ [1], (Fig.1). The shell is weakened by the surface crack located in the crossection $\alpha=0$ or $\beta=0$ and is under the forces and moments symmetric about the crack. $2\,l_0$ and $2d$ are the length and depth of the crack, respectively, $2h$ and $R$ are the thickness of the shell and radius of its medium surface, respectively, $\gamma$ is a coordinate normal to the medium surface.

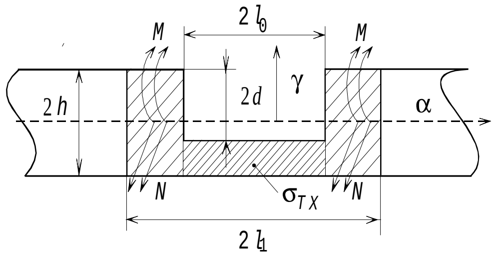


**Figure 1:** Scheme of surface crack.

It is assumed that on the crack extension in depth, i.e. on the domain $x\in\ ]-x_0, x_0[$ and $-h\le\gamma\le h-2d$, the constant stresses $\sigma_{TX}$ are acting (here $x=\alpha,\beta$ corresponds to the crack location, $x_0=l_0/R$, $\sigma_{TX}$ is yield point for the shell material perpendicular to the crack surface direction). In the plastical zones in the crack extension $x_0<|x|<x_1$ ($x_1=l_1/R$) the unknown normal force $N$ and bending moment $M$ act.
In addition, $N$ and $M$ satisfy the corresponding plasticity condition, e.g., the Treska condition in the form of plastic hinge [2]

$$\left(\frac{N(x)}{2h\sigma^*(x)}\right)^2+\frac{|M(x)|}{h^2\sigma^*(x)}=1, \qquad x_0\le|x|\le x_1 \tag{1}$$

For the case of ideal elastoplastic material $N(x)$ and $M(x)$ are assumed to be constant , $\sigma^*(x)=\sigma_{TX}$ [2 ]
For reinforced material [3]

$$\begin{aligned} N(x)&=P\left[(1-m^*)(|x|-x_0)(x_1-x_0)+m^*\right]\\ M(x)&=H\left[(1-m^*)(|x|-x_0)(x_1-x_0)+m^*\right]\\ \sigma^*(x)&=(\sigma_{Tx}-\sigma_{Bx})(|x|-x_0)(x_1-x_0)+\sigma_{TX}\end{aligned} \tag{2}$$

Here $m^*=\sigma_{BX}/\sigma_{TX}$, $\sigma_{BX}$ is the strength limit of material perpendicular to the crack surface direction, $P,H$ are such unknown constants , that condition Eqn.1 is satisfied.
But under such approach such condition is not fulfilled for normal stresses

$$\sigma_n(x_0-0,\gamma^*)=\sigma_n(x_0+0,\gamma^*) \tag{3}$$

for any point (0, $x_0$, $\gamma^*$). This is impossible, because always may be found $\gamma^*\in[-h,h-2d]$, for which $\sigma_n(x_0-0,\gamma^*)=\sigma_{TX}$, and $\sigma_n(x_0+0,\gamma^*)<\sigma_{TX}$.
Assume $\sigma_n=\sigma_{TX}$, at $x=\pm(x_0+0)$ through the whole shell thickness, i.c., for all values of $\gamma$. But then through the whole thickness $N(x_0\pm0)=2h\,\sigma_{TX}$, $M(x_0\pm0)=0$. Since the line of unit fibre under the known plasticity conditions for thin shells is constant, in the plastical zone $x_0<|x|<x_1$ the bending moment is absent. It means, the stresses diagram along the plastical zones must change the coordinate of unit fibre. Therefore, the

condition Eqn.1 will be presented in the form:

$$\frac{N(\tau)}{2h} = -k_m \xi_n(\tau)\sigma^*(\tau), \qquad \frac{M(\tau)}{h^2} = -k_m \xi_n(\tau)\sigma^*(\tau)\left[1-\xi_n^2(\tau)\right] \quad \tau_0 \le \tau \le 1, \tag{4}$$

Here $\tau = x/x_1$ ; $\tau_0 = x_0/x_1$ ;$k_m$=sgn$H$ ; $\xi_n = \gamma_n/h$ ; $\gamma_n$ is the unit fibre coordinate.

$$\sigma^*(\tau) = \sigma_{TX}\left(\nu_2|\tau| + \nu_1\right), \quad \nu_1 = \frac{m^* - \tau_0}{1-\tau_0}, \quad \nu_2 = \frac{1-m^*}{1-\tau_0}, \tag{5}$$

Let $\gamma_n$ change along the plastical zones linearly.

$$\xi_n(\tau) = k_m\left(-1 + (1-p^*)\frac{|\tau| - \tau_0}{1-\tau_0}\right), \quad \tau_0 \le \tau \le 1 \quad p^* = \frac{P}{2h\sigma_{TX}}, \tag{6}$$

Substituting Eqn.6 in Eqn.4 and taking into account Eqn.5, we shall obtain

$$\frac{N(\tau)}{\sigma_{TX}} = n_0 + n_1\tau + n_2\tau^2, \quad \frac{M(\tau)}{h^2\sigma_{TX}} = k_m\omega\left(m_0 + m_1|\tau| + m_2\tau^2 + m_3|\tau|^3\right), \tag{7}$$

Here

$$\begin{gathered} n_0 = \nu_1(1+\omega\tau_0) \quad n_1 = \nu_2(1+\omega\tau_0) - \omega\nu_1, \quad n_2 = -\omega\nu_2 \\ m_0 = -\nu_1\tau(2+\omega\tau_0) \quad m_1 = 2\nu_1(1+\omega\tau_0) - \nu_2\omega\tau_0(2+\omega\tau_0), \\ m_2 = -\nu_1\omega + 2\nu_2(1+\omega\tau_0) \quad m_3 = n_2, \quad \omega = \left(1-p^*\right)/\left(1-\tau_0\right). \end{gathered} \tag{8}$$

Thus, from Eqns5,6 follows, that for $\tau = \pm\tau_0$ the stresses along all shell thickness are equal to $\sigma_{TX}$, because $N(x_0+0) = 2h\,\sigma_{TX}$, $M(x_0+0) = 0$ and this means that the condition from Eqn.3 for any $\gamma \in [-h, -h+2d]$ is satisfied. $M(\pm 1) = H$. Subject to the sign of bending moment, two cases $\xi_n(\pm\tau_0)=1$ or $\xi_n(\pm\tau_0)=-1$ are possible.
So, in order the normal stresses in the plastical zones be continuos, it is necessary, that a coordinate of the unit fibre change even under the linear law Eqn.6, and the normal force and bending moment are, respectively, quadratic and cube polynomials from the coordinate $\tau$.
Hence, within the scope of this model the three dimensional problem on determination of the stressed state and limit equilibrium of the shell with a surface crack of length $2l_0$ is reduced to the two-dimensional one of elasticity theory for a shell with a through crack of unknown length $2l_1$, on the edge of which the following conditions

$$N_k(x) = \begin{cases} N_l - N_k^0(x), & |x| < x_0, \\ N(x) - N_k^0(x), & x_0 \le x \le x_1, \end{cases} ; \; M_k(x) = \begin{cases} M_l - M_k^0(x), & |x| < x_0, \\ M(x) - M_k^0(x), & x_0 \le x \le x_1, \end{cases} \tag{9}$$

are satisfied. Here $N_l$, $M_l$ are normal force and bending moment, respectively, which are the reaction of material on the discontinuity of inner bonds over the crack. According to our assumption about stresses in this zones they are determined according to the formulas

$$N_l = 2\sigma_{Tx}(h-d), \qquad M_l = 2\sigma_{Tx}(h-d)d \tag{10}$$

$N_k^0, M_k^0$ are normal force and bending moment, respectively, in the shell without crack, caused by an outer load ($k$=1 for a circumferential crack, $k$=2 for a longitudinal crack)

## INTEGRAL EQUATIONS FOR OUR PROBLEM

On the basis of distortion method a system of resolving nonhomogeneous differential equations of the tenth order, which takes into account the presence of displacements jumps and rotation angles due to the crack, is written down for solving the obtained elastic problem. The equations of Tymoshenko-type shell theory are taken as the input ones. Using the $2\pi$-periodic (along a circumferential coordinate) fundamental solution to this system [4,5], the efforts and moments integral representation in terms of unknown jumps of generalised displacements is written. Satisfying the boundary conditions on the crack opposite edges Eqn.9, the problem is reduced to a system of singular integral equations with unknown limits of integration ($l_1$ under the problem formulation is unknown) and the discontinuous right-hand parts, which include the unknown values of efforts and moments, acting in the plastic zones In the case of symmetric loading the system is such

$$\sum_{i=1}^{2}\int_{-1}^{1}\varphi_i(u)\left\{\frac{a_{ij}}{t-u}+l_1 K_{ij}[l_1(u-t)]\right\}du = c_j f_j(l_1 t),\quad j=1,2,\quad |t|<1 \tag{11}$$

where

$$\varphi_1(x)=\frac{d}{dx}[u_x(x)],\qquad \varphi_2(x)=\frac{d}{dx}[\gamma_x(x)],\qquad t=\frac{x}{l_1} \tag{12}$$

$$f_1(x)=N_k(x),\qquad f_2(x)=M_k(x)$$

The expressions for $\alpha_{ij}$ $K_{ij}$ $c_j$ for orthotropic cylindrical shells with crack are given in [5], for isotropic and transversally-isotropic shells are given in [1,3] $[u_x(x)]$, $[\gamma_x(x)]$ are the unknown jumps of normal displacement and rotation angle under transition over the crack line. The expressions for determination of *N(x)* and *M(x)* are contained in conditions Eqn.9, what means in the right-hand part of the system of integral equations Eqn.11. Thus, in these equations besides the unknown integral limits $l_1$, the values *P*,*H* are also unknown. So, the system Eqn.11 is complimented by the plasticity condition Eqn.1 and also by the conditions of boundedness of normal force and bending moment near the crack, i.e. the corresponding stress-intensity factors must be equal to zero.

$$K_N(x_1)=0,\qquad K_M(x_1)=0 \tag{13}$$

On integrating the solution obtained and having substituted it into formula

$$\delta(x,\ \gamma)=[u_x(x/x_1)]+\gamma[\gamma_x(x/x_1)],\quad |x|<x_1,\quad |\gamma|<h \tag{14}$$

we obtain the relation for determination of opening of the crack edges at any point. This relation after substitution of critical value of the crack opening $\delta_c$ for $\delta(x,\gamma)$ becomes a criterion equation which establishes connection between the applied load, crack dimensions, physical-mechanical and geometric parameters of the shell under conditions of limit equilibrium state.

## THE SOLUTION OF INTEGRAL EQUATIONS

We shall note , that the right-hand sides of the system Eqn.11 are discontinuous functions. The direct methods for such system solution, as it was shown by the numerical tests [1] gives a large error at the discontinuity point. Since, we are interested in the crack opening at this point.
The solution of the system will be presented in the form [6]

$$\varphi_i(t) = h_i(t) + \psi_i(t), \qquad i = 1,\ 2\,, \tag{15}$$

where $h_i(t)$ is the solution for a corresponding canonical singular integral equations with the discontinuous right-hand parts.

$$\int_{-1}^{1} \frac{h_i(t)}{t-\xi}\,dt = c_i\ f_i(\xi), \qquad |\xi| < 1, \qquad i = 1,\ 2\,. \tag{16}$$

This system may be solved using the inversion formula for Cauchy type integrals. Thus, when $N_k^0,\ M_k^0$ are constant, $h_i(t)$ is such

$$h_i(t) = c_i\left[H_0^{(i)}(t) + \Delta_i\ H_1(t) + b_{2i}k^* k_m\,\omega^2\ H_2(t)\right] \Big/ \left(\pi^2\,\Delta\sqrt{1-t^2}\right) \tag{17}$$

where

$$\begin{gathered}
H_0^{(i)}(t) = D_i^0\,t - D_i\ Z(t) + b_{i1}\,Q_0(t), \qquad H_i(t) = \tau_0 Q_0(t) - Q_i(t), \quad i = 1,\ 2\,, \\
Q_j(t) = \nu_1\ I_j(t) + \nu_2\ I_{j+1}(t), \qquad j = 0,\ 1,\ 2, \qquad I_0(t) = -Z(t)\,, \\
I_1(t) = -t\left[2q_0 + \sqrt{1-t^2}\ L_2(t)\right], \quad I_2(t) = -t\left[(\tau_0\,q_0 - \vartheta^*) + t\sqrt{1-t^2}\ L_1(t) + 2\,t^2\vartheta^*\right], \\
I_3(t) = t\left[\frac{2}{3}q_0^3 - 2\,q_0\,t^2 - t^2\sqrt{1-t^2}\ L_2(t)\right], \qquad \Delta_i = \omega\left[b_{1i} - b_{2i}k^* k_m(2 + \omega\,\tau_0)\right], \\
\Delta = a_{11}\,a_{22} + a_{12}^2, \qquad D_i^0 = F_R^{(i)}(N_\varepsilon^0,\ M_\varepsilon^0), \qquad D_i = F_R^{(i)}(N,\ M)\,, \\
F_R^{(i)}(X,\ Y) = b_{1i}\frac{X}{2h\sigma_T} + b_{2i}\frac{k^* Y}{h^2\,\sigma_T}, \qquad b_{ij} = \begin{cases} a_{ij}, & i = j, \\ -a_{ji}, & i \neq j, \end{cases} \qquad i,\ j = 1,\ 2\,, \\
Z(t) = 2\vartheta^* t + \sqrt{1-t^2}\ L_1(t), \qquad \vartheta^* = \arccos\tau_0\,, \\
L_1(t) = \ln\left|\frac{\tau_0\sqrt{1-t^2} - t\sqrt{1-\tau_0^2}}{\tau_0\sqrt{1-t^2} + t\sqrt{1-\tau_0^2}}\right|, \qquad L_2(t) = \ln\left|\frac{\sqrt{1-t^2} - \sqrt{1-\tau_0^2}}{\sqrt{1-t^2} + \sqrt{1-\tau_0^2}}\right|
\end{gathered} \tag{18}$$

.

Substituting Eqn.15 in Eqn.11 and taking into account Eqn.16 we obtain a system of singular integral equations for determining the function $\psi_i(t)$ . This system is the same as Eqn.11, where $\varphi_i(t)$ is changed by $\psi_i(t)$ , and $f_j(t)$ is changed by $f_j^*(t)$, where

$$f_j^*(t) = x_1\int_{-1}^{1}\sum_{i=1}^{2} h_i(\xi)\,K_{ij}[x_1(t-\xi)]\,d\xi \tag{19}$$

Taking into account , that $h_i(\xi)$ are determined by the unknown variables $H,\ P$ , the Eqn.19 is written in the form

$$f_j^*(t) = g_j^0(t) + \sum_{p=1}^{2} G_p\ g_j^{(p)}(t), \qquad j = 1,\ 2 \tag{20}$$

where $G_1 = N/(2h)$; $G_2 = M/h^2$; $g_j^{(p)}$, $p = 0, 1, 2$ are expressed by the functions $h_i(t)$ and $K_{ij}(z)$. According to Eqn.20 the functions $\psi_i(t)$ will be presented in the form of linear combination

$$\psi_i(t) = \psi_i^0(t) + \sum_{p=1}^{2} G_p \psi_i^{(p)}(t), \qquad i = 1, 2 \tag{21}$$

Every couple $\psi_i^{(p)}(t)$, $i = 1, 2$; $p = 0, 1, 2$ is the solution of a system of singular integral equation of type from Eqn.11 with the right-hand part $g_j^{(p)}$, and it satisfies the conditions

$$\int_{-1}^{1} \psi_i^{(p)}(t)\, dt = 0, \qquad p = 0, 1, 2 \tag{22}$$

The system of integral equations from Eqn.11 taking into account Eqn.20 is constructed by the mechanical quadratures method [5]. This allows to reduce its solution to the system of linear algebraic equations. But, the unknown length of plastical zone is contained in these equations nonlinearly. So the procedure for solving these equations is such. In some way $x_1$ is chosen, then the system of integral equations is solved for each $p$=0,1,2. From condition Eqn.13 we determine $P$,$H$ and test the plasticity condition Eqn.1. If this condition is fulfilled to a prescribed accuracy, the problem is solved, and if not, the value $x_1$ is changed, and the procedure is repeated. As it was said, that the crack opening at any point is found by the formula from Eqn.14.

## NUMERICAL RESULTS

Numerical analysis for different values of $d$ for a transversally-isotropic shell with a longidudial crack is carried out. The shell is under inner pressure $p$. In Fig.2 the graphics of functions for relative values of the crack front opening $\delta^* = \delta(0, h - 2d)\, E/(l_0 \sigma_{Tx})$ versus the parameter $n^0$ , which characterizes the outer loading are presented, $n^0 = pR/(2h\sigma_{Tx})$. The computation is made for such values of parameters $h/R = 0.01$, $\nu = 0.3$, $d/h = 0.6$, The curves (1), (2) are calculated for the crack length $l_0 = 0.1R$ , the curves (3), (4) for the $l_0 = 0.2R$.

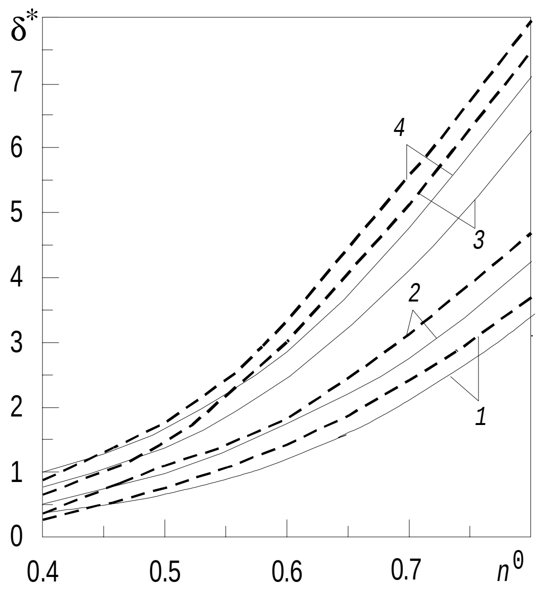


**Fig.2:** Crack front opening versus crack length , loading and shear module.

▪▪▪▪▪▪▪▪▪▪▪▪ corresponds $\sigma_n(a) = \text{const},\ x_0 < |x| < x_1$.

——— corresponds to the results obtained within the scope of proposed model

In addition, the curves (1), (3) correspond to the isotropic shell ($E/G' = 2.6$) and (2), (4) correspond to the transversally-isotropic one ($E/G' = 20$). $E/G'$ is the ratio of the elasticity module to the shear module in the area elements, perpendicular to the medium surface. When $E/G'$, $n_0$, $l_0$ increase, the deviation between the results for different models will be increase too. Thus, at $n^0 = 0.8$, $l_0 = 0.2R$ this differences exceeds 20%. When the crack depth increases the difference between the results obtained by the two models will decrease. It should be noted also that when the crack is increasing, the anisotropy influence on its opening is decreases.

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