THREE-DIMENSIONAL AXISYMMETRIC VIBRATIONS OF ANISOTROPIC LAMINATED CYLINDERS

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Free vibration of arbitrary laminated, anisotropic cylindrical shells with finite lengths is studied, using elasticity approach. The coupled partial differential equations are reduced to ordinary differential equations(ODE) by choosing the solution composed of trigonometric functions along the axial direction. Through dividing each layer into thin laminae, the variable coefficients in ODEs become constant. Combining all exact solutions obtained by means of appropriate continuity conditions, the corresponding solution of the exact governing equations is successively approached. Numerical examples are also presented.

INTROUDUCTION

The mathematical complexity in analyzing three-dimensional elasticity equations usually makes exact solutions difficult to obtain. However certain problems in which a three-dimensional approach can be used still exist. Most of these problems can be solved by assuming the solution to be composed of trigonometric functions in the axial and circumferential directions.The solution for the resulting ODEs can be obtained by introducing the displacement potential function. Usually this method is used with isotropic and transversely isotropic material, whereas the Frobenius method is used with orthotropic materials.
Although the aforementioned three-dimensional elasticity approach provides exact solution, considerable mathematical complexity prevents more general problem from being solved. As a consequence, an approximate elasticity approach under the assumption of

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$h_k/R_k \ll 1$ (where h_k and R_k denote the thickness and mean radius of the k th.lamina) was suggested by Soong(1). The advantage of this assumption is that the the ODEs with variable coefficients can be reduced to ODEs with constant coefficients that can be solved exactly. Based on this assumption recently the authors have considered free vibrations of laminated anisotropic hemispherical shells(2).

In this paper solution based on elasticity equations is presented for axisymmetric vibrations of arbitrarily laminated,anisotropic cylindrical shells of finite length.

PROBLEM FORMULATION

Consider a laminated composite hollow cylindrical shell of length L with M constituent orthotropic laminae. The mean radius and the thickness of layers are denoted by R_k and h_k , $k=1,2,..M$ respectively. The material axes of any orthotropic layer are not necessarily aligned with the x,and θ directions. Hence The constitutive equations of a layer are as follows:

$$\begin{Bmatrix} \sigma_x \\ \sigma_\theta \\ \sigma_r \\ \tau_{r\theta} \\ \tau_{xr} \\ \tau_{x\theta} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ C_{12} & C_{22} & C_{23} & 0 & 0 & C_{26} \\ C_{13} & C_{23} & C_{33} & 0 & 0 & C_{36} \\ 0 & 0 & 0 & C_{44} & C_{45} & 0 \\ 0 & 0 & 0 & C_{44} & C_{55} & 0 \\ C_{16} & C_{26} & C_{36} & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_\theta \\ \varepsilon_r \\ \gamma_{r\theta} \\ \gamma_{xr} \\ \gamma_{x\theta} \end{Bmatrix} \tag{1}$$

The equations of motion in terms of displacement components in cylindrical coordinates for a material with constitutive Eq.(1) are

$$C_{33}\frac{\partial^2 u_r}{\partial r^2}+C_{33}\frac{\partial u_r}{r\partial r}+C_{55}\frac{\partial^2 u_r}{\partial x^2}-C_{22}\frac{u_r}{r^2}+(C_{36}-C_{45}-C_{26})\frac{\partial u_\theta}{r\partial x}+(C_{36}+C_{45})$$
$$\frac{\partial^2 u_\theta}{\partial r\partial x}+(C_{13}-C_{12})\frac{\partial u_x}{r\partial x}+(C_{13}+C_{55})\frac{\partial^2 u_x}{\partial x\partial r}=\rho\frac{\partial^2 u_r}{\partial t^2},$$
$$(2C_{45}+C_{26})\frac{\partial u_r}{r\partial x}+(C_{45}+C_{36})\frac{\partial^2 u_r}{\partial r\partial x}+C_{44}\frac{\partial^2 u_\theta}{\partial r^2}+C_{44}\frac{\partial u_\theta}{r\partial r}-C_{44}\frac{u_\theta}{r^2}$$
$$+C_{66}\frac{\partial^2 u_\theta}{\partial x^2}+C_{45}\frac{\partial^2 u_x}{\partial r^2}+2C_{45}\frac{\partial u_x}{r\partial x}+C_{16}\frac{\partial^2 u_x}{\partial x^2}=\rho\frac{\partial^2 u_\theta}{\partial t^2},$$
$$(C_{12}+C_{55})\frac{\partial u_r}{r\partial x}+(C_{13}+C_{55})\frac{\partial^2 u_r}{\partial r\partial x}+C_{45}\frac{\partial^2 u_\theta}{\partial r^2}+C_{16}\frac{\partial^2 u_\theta}{\partial r^2}+C_{11}\frac{\partial^2 u_x}{\partial x^2}$$
$$+C_{55}\left(\frac{\partial^2 u_x}{\partial r^2}+\frac{\partial u_x}{r\partial r}\right)=\rho\frac{\partial^2 u_x}{\partial t^2}, \tag{2}$$

The coefficients of the above equations are functions of variable r which makes the solution formidable. To circumvent this difficulty, the following change of variable are used [1]

$$\frac{1}{r} = \frac{1}{R_k}(1-\eta_k), \quad \frac{1}{r^2} = \frac{1}{R_k^2}(1-2\eta_k), \quad \eta_k = \frac{r}{R_k} - 1 \qquad (3)$$

Eq.(2) in terms of the new variables become

$$C_{33}^k\left[\frac{\partial^2 u_r}{\partial \eta_k^2} + \frac{\partial u_r}{\partial \eta_k}\right] + C_{55}^k R_k^2 \frac{\partial^2 u_r}{\partial x^2} - C_{22}^k u_r + R_k\left[(C_{36}^k - C_{45}^k - C_{26}^k)\frac{\partial u_\theta}{\partial x}\right.$$

$$\left. + (C_{36}^k + C_{45}^k)\frac{\partial^2 u_\theta}{\partial x \partial \eta_k} + (C_{13}^k - C_{12}^k)\frac{\partial u_x}{\partial x} + (C_{13}^k + C_{55}^k)\frac{\partial u_x}{\partial x \partial \eta_k}\right] = \rho^k R_k^2 \frac{\partial^2 u_r}{\partial t^2}$$

$$R_k\left[(2C_{45}^k + C_{26}^k)\frac{\partial u_r}{\partial x} \quad (C_{45}^k + C_{36}^k)\frac{\partial^2 u_r}{\partial x \partial \eta_k}\right] + C_{44}^k\left(\frac{\partial^2 u_\theta}{\partial \eta_k^2} + \frac{\partial u_\theta}{\partial \eta_k}\right.$$

$$\left. - u_\theta\right) + R_k^2 C_{66}^k \frac{\partial^2 u_\theta}{\partial x^2} + C_{45}^k \frac{\partial^2 u_x}{\partial \eta_k^2} + 2C_{45}^k \frac{\partial u_x}{\partial \eta_k} + R_k^2 C_{16}^k \frac{\partial^2 u_x}{\partial x^2} = \rho^k R_k^2 \frac{\partial^2 u_\theta}{\partial t^2}$$

$$R_k\left[(C_{12}^k + C_{55}^k)\frac{\partial u_r}{\partial x} + (C_{13}^k + C_{55}^k)\frac{\partial^2 u_r}{\partial x \partial \eta_k}\right] + C_{45}^k \frac{\partial^2 u_\theta}{\partial \eta_k^2} + R_k^2 C_{16}^k \frac{\partial^2 u_x}{\partial x^2}$$

$$+ R_k^2 C_{11}^k \frac{\partial^2 u_\theta}{\partial x^2} + C_{55}^k\left(\frac{\partial^2 u_x}{\partial \eta_k^2} + \frac{\partial u_x}{\partial \eta_k}\right) = \rho^k R_k^2 \frac{\partial^2 u_x}{\partial t^2} \qquad (4)$$

In the derivation of the above equations, we made use of the approximation $1 + \eta_k \cong 1$

For simply supported boundary conditions

$$\sigma_x(o,\eta) = \sigma_x(L,\eta) = o \quad , \quad u_r(o,\eta) = u_r(L,\eta) = o \qquad (5)$$

The inner and outer surfaces are tration free. Thus

$$\sigma_r = \tau_{r\theta} = \tau_{xr} = o \qquad (6)$$

Moreover, the conditions of continuity of displacement and interlaminar stresses are

$$u_r^k\left(x, \frac{h_k}{2R_k}\right) = u_r^{k+1}\left(x, -\frac{h_{k+1}}{2R_{k+1}}\right) \qquad u_\theta^k\left(x, \frac{h_k}{2R_k}\right) = u_\theta^{k+1}\left(x, -\frac{h_{k+1}}{2R_{k+1}}\right)$$

$$u_x^k\left(x, \frac{h_k}{2R_k}\right) = u_x^{k+1}\left(x, -\frac{h_{k+1}}{2R_{k+1}}\right) \qquad \sigma_r^k\left(x, \frac{h_k}{2R_k}\right) = \sigma_r^{k+1}\left(x, -\frac{h_{k+1}}{2R_{k+1}}\right)$$

$$\tau_{r\theta}^k\left(x, \frac{h_k}{2R_k}\right) = \tau_{r\theta}^{k+1}\left(x, -\frac{h_{k+1}}{2R_{k+1}}\right) \qquad \tau_{xr}^k\left(x, \frac{h_k}{2R_{k+}}\right) = \tau_{xr}^{k+1}\left(x, -\frac{h_{k+1}}{2R_{k+1}}\right) \qquad (7)$$

Successive Approximation Solution

The solutions to equations (4) which identically satisfies the boundary conditions on the two ends are considered as

$$u_r^k=\Sigma_{m=1}^{\infty}\sin p_m x A_r(\eta)e^{i\omega t}\ ,\quad u_\theta^k=\Sigma_{m=1}^{\infty}\cos p_m x A_\theta(\eta)e^{i\omega t}$$
$$u_x^k=\Sigma_{m=1}^{\infty}\cos p_m x A_x(\eta)e^{i\omega t}\qquad P_m=\frac{m\pi}{L}\tag{8}$$

The substitution of (8) into (4) yields a homogenous system of ordinary differential equations which the solution to those are

$$A_r(\eta_k)=u_r^* e^{\lambda\eta_k}\ ,\ A_\theta(\eta_k)=u_\theta^* e^{\lambda\eta_k},\ A_x(\eta_k)=u_x^* e^{\lambda\eta_k}\tag{9}$$

Where u_r^* , u_θ^* and u_x^* are the unknown coefficients.
Upon inserting solutions(9) into Eqs.(8) a system of homogenous algebric equations are obtained which may be written in matrix form as

$$[A]\{U^*\}=0\qquad \text{where}\qquad \{U^*\}^T=\{u_r^*\ u_\theta^*\ u_x^*\}\tag{10}$$

The condition for Eq.(10) to have nontrival solution is that the determinate of matrix A should vanish. This leads to a sixth order algebric equation

$$A\lambda^6+B\lambda^5+C\lambda^4+D\lambda^3+E\lambda^2+F\lambda+G=0\tag{11}$$

The displacement components may be obtained which are functions of natural frequency ω . By substituting the roots of Eq.(11) into (8) lead to:

$$u_r^k=\Sigma_{m=1}^{\infty}\Sigma_{j=1}^{6}K_{mj}^k e^{\lambda_j\eta}\sin(P_m x)e^{i\omega t}$$
$$u_\theta^k=\Sigma_{m=1}^{\infty}\Sigma_{j=1}^{6}P_{mj}^k K_{mj}^k e^{\lambda_j\eta}\cos(P_m x)e^{i\omega t}$$
$$u_x^k=\Sigma_{m=1}^{\infty}\Sigma_{j=1}^{6}Q_{mj}^k K_{mj}^k e^{\lambda_j\eta}\cos(P_m x)e^{i\omega t}\tag{12}$$

Where P_j^k and Q_j^k are function of ω.
Substituting (12) into the traction free conditions (6) and continuity requirements (7) leads to a system of 6M homogeneous algebric equations which may be represented as

$$[H]\{K\}=0\tag{13}$$

The vector {K} is the mode shape.The components of [H] which are 6Mx6M matrices are functions of ω. From (13) we have

$$|H|=0\tag{14}$$

Eqs.(11) and (14) should be solved simultanously by the successive approximation procedure to obtain the first few natural frequencies.

RESULTS

The layers of laminated cylinder are constructed by graphite-epoxy material having the properties as

$E_1/E_2=15, \quad G_{23}/E_2=0.342, \quad G_{12}/E_2=G_{13}/E_2=0.28, \quad \nu_{12}= 0.4$

Fig.(1) exhibit the variations of lowest natural frequency parameter ($\hat{\omega}$) of antisymmetric (45/-45) and symmetric (45/-45/-45/45) angle-ply with respect to the length to thickness ratio (L/h).The radius to thickness ratio is R/h = 5. As it is noticed the natural frequencies of symmetric angle-ply are generally higher than their antisymmetric counterpart in the entire range of L/h considered, and that the corresponding curves have stiffer slopes, especially in the thicker shell regime. This is due to the effect of bending-streching type coupling that charaterizes an antisymmetric laminate. Fig.(2) show the mode shapes corresponding to the first frequency parameters of a relatively thick (h/R= 0.5 ,mR/L=1) two - layered (45/0) angle ply. As expected the laminate is constrained in the axial direction within the outer ply, with little constraint in the inner ply. Due to much higher axial reinforcement, the axial displacements are smaller in the interior of outer ply.

SYMBOLS USED

C_{ij} = stiffness elastic constants

L = length of cylindrical shell

u_r = radial displacement

u_x = axial displacement

u_θ = circumferential displacement

REFERENCES

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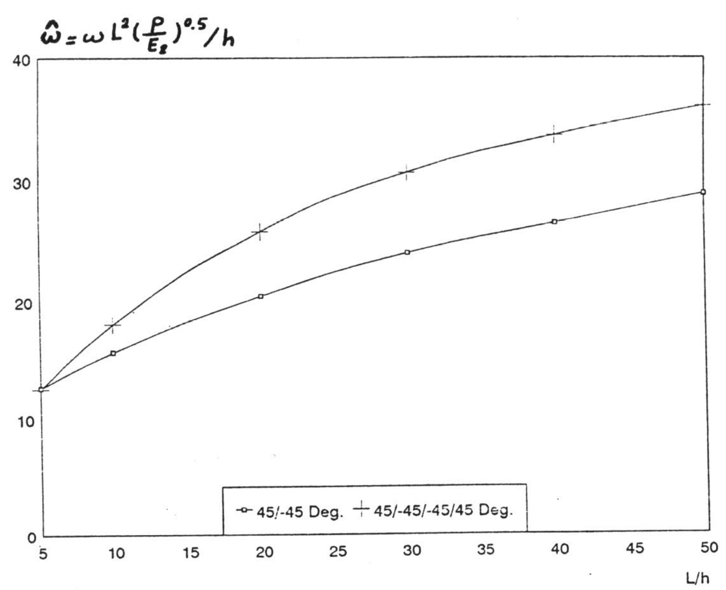

Figure 1 Variation of lowest natural frequency parameter

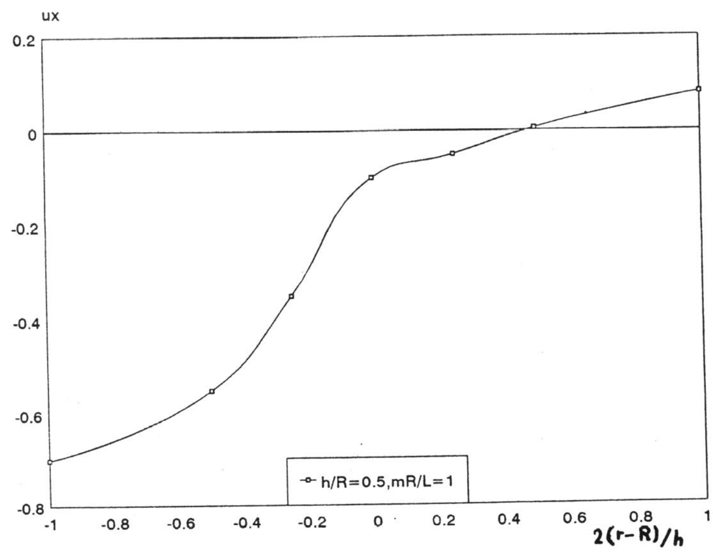

Figure 2 Variation of axial mode