SINGULARITY FIELDS OF A CRACK MEETING AN INCLINED INTERFACE OF TWO ELASTIC MATERIALS AND NUMERICAL RESULTS

J. LI, X.B. ZHANG and N. RECHO *

The aim of this paper is to study the behaviour of a crack terminating at an arbitrary angle with an interface of two elastic materials. The asymptotic fields of stress and displacement near the crack tip are found by using the eigenfunction expansion method. A numerical method to determine the stress intensity factors is developed. The found asymptotic fields are verified numerically by the use of the finite element method. These fields are also in good agreement with the analytical solutions for the particular cases such as crack perpendicular to the interface given by Cook and Erdogan (1). Finally, the crack propagation is discussed.

INTRODUCTION

Most of the experimental and theoretical investigations for bimaterial crack problem has focused on the few particular cases of crack orientations such as crack lying along, or perpendicular to, the interface. However, cracks advancing or terminating at arbitrary angles with an interface between two materials may also be found very often. So far, only few works in literature have dealt with this crack configuration (2). The purpose of this paper is to establish the asymptotic fields of such a bimaterial crack. The stress intensity factors can then be determined to predict the crack propagation.

STRESS SINGULARITIES AND ASYMPTOTIC FIELDS

Consider a semi-infinite crack terminating at an incidence angle $\theta_0 (0 \leq \theta_0 \leq \pi/2)$ formed between the crack and the interface of two homogeneous, isotropic elastic materials (fig. 1). Material 1 occupies region 1 ($\theta_0 \leq \theta \leq \pi$) and region 3 ($-\pi \leq \theta \leq \theta_0 - \pi$), with Young's modulus E_1 and Poisson's ratio ν_1. Material 2 occupies region 2 ($\theta_0 - \pi \leq \theta \leq \theta_0$), with material constants E_2 and ν_2. By using Williams eigenfunction expansion method (3), the Airy function is supposed as follows:

*Laboratoire d'Etudes et de Recherches en Mécanique des Structures
University B.P. of Clermont-Fd - Av. Aristide Briand 03100 Montluçon France

$$U^{(i)}(r,\theta)=r^{\lambda+1}F^{(i)}(\theta) \quad (1)$$

where i = 1, 2 or 3 for each region respectively. The singularity exponent λ could be real or complex . By introducing the following boundary conditions of the problem: the free surface at the crack lips and the continuity of stresses and displacements at the interface, a system of 12 linear equations with 12 undetermined constants is obtained. To have a non-trivial solution of these linear equations, one has to verify that the determinant of the operator matrix $[L(\lambda)]$ is equal to zero:

$$\det[L(\lambda)]=0 \quad (2)$$

The Newton-Raphson method is used to determine the eigenvalues which lead, when substituted in equation (1), to singular stress fields. It has been found that the eigenvalues take different forms. This depends not only on the incidence angle θ_0 between the crack and the interface, but also on a characteristic angle θ_t which varies with the ratio of the two material constants. Figure 2 shows the variation of θ_t with the ratio of the two material shear modulus μ_1/μ_2. Figure 3 shows the variation of the form of λ for $\mu_1/\mu_2 = 0.2$ and 5. According to the form of the eigenvalues λ, three different cases can be distinguished :

(i) One pair of real values case

When $\mu_1/\mu_2 \neq 1$ and $\theta_t<\theta_0<90°$, one pair of real eigenvalues, λ_1 and λ_2, is found. In the case of the crack lying in the less stiff material ($\mu_1/\mu_2<1$), we have $0.5<\lambda_1<\lambda_2<1$. On the contrary ($\mu_1/\mu_2>1$), one finds $0\leq\lambda_1<\lambda_2<0.5$.

The displacement and stress fields can be defined in each region as follows:

$$u_r^{(i)} = \frac{1}{2\mu}\left\{r^{\lambda_1} k_1.\tilde{u}_{r_1}^{(i)}(\theta)+r^{\lambda_2} k_2.\tilde{u}_{r_2}^{(i)}(\theta)\right\},\quad u_\theta^{(i)} = \frac{1}{2\mu}\left\{r^{\lambda_1} k_1.\tilde{u}_{\theta_1}^{(i)}(\theta)+r^{\lambda_2} k_2.\tilde{u}_{\theta_2}^{(i)}(\theta)\right\}$$

$$\sigma_{rr}^{(i)} = k_1 r^{\lambda_1-1}.\tilde{\sigma}_{rr_1}^{(i)}(\theta)+k_2 r^{\lambda_2-1}.\tilde{\sigma}_{rr_2}^{(i)}(\theta)$$

$$\sigma_{\theta\theta}^{(i)} = k_1 r^{\lambda_1-1}.\tilde{\sigma}_{\theta\theta_1}^{(i)}(\theta)+k_2 r^{\lambda_2-1}.\tilde{\sigma}_{\theta\theta_2}^{(i)}(\theta)$$

$$\sigma_{r\theta}^{(i)} = k_1 r^{\lambda_1-1}.\tilde{\sigma}_{r\theta_1}^{(i)}(\theta)+k_2 r^{\lambda_2-1}.\tilde{\sigma}_{r\theta_2}^{(i)}(\theta) \quad (3)$$

where $\tilde{u}_{r_n}^{(i)}(\theta), \tilde{u}_{\theta_n}^{(i)}(\theta), \tilde{\sigma}_{rr_n}^{(i)}(\theta), \tilde{\sigma}_{\theta\theta n}^{(i)}(\theta)$ and $\tilde{\sigma}_{r\theta n}^{(i)}(\theta)$ are θ-dependent functions (4) and the index n = 1 or 2 corresponding to the two eigenvalues respectively. k_1 and k_2 are the factors dependent on the remote loading conditions like the conventional stress intensity factors K_I and K_{II}. However, they have not exactly the same signification as K_I and K_{II} because they don't correspond respectively to the two fracture modes but to the two eigenvalues λ_1 and λ_2.

(ii) Single real value case

When $\theta_0 = 90°$ and $\theta_0 = \theta_t$, a single real eigenvalue λ is found. This is also true for the homogeneous material case ($\mu_1/\mu_2=1$) in which $\lambda=0.5$. In this case, by substituting $\lambda=\lambda_1=\lambda_2$ into equation (3) and by using symmetrical or anti-symmetrical properties related to each fracture mode, the displacement and stress asymptotic fields can be written as a function of the conventional stress intensity factors K_I and K_{II} (4). One can then separate the two fracture modes easily.

(iii) One pair of complex conjugate values case

When $\mu_1/\mu_2 \neq 1$ and $0° \leq \theta_0 < \theta_t$, the eigenvalues become complex: $\lambda_{1,2} = \lambda_r \pm i\lambda_j$. All the components of its eigenvector become complex. $\lambda_r<1$ and hence leads to a stress singularity. The asymptotic stress and displacement fields may be represented by the real part of the equation (3). If $\theta_0=0$, the same results as that obtained by Williams (3) for homogeneous material case is found: $\lambda_r = 0.5$.

Since λ is complex, so, $r^{\lambda} = r^{\lambda_r}[\cos(\lambda_j \ln r) + i\sin(\lambda_j \ln r)]$. It means that the behaviour of stress and displacement fields near the crack-tip is oscillatory with stress being bounded by $r^{\lambda_r - 1}$.

The factors k_1 and k_2 can also be related to the conventional stress intensity factors K_I and K_{II} by their definition.

NUMERICAL RESULTS

As discussed above, the factors k_1 and k_2 have not exactly the same signification as K_I and K_{II} for homogeneous problem. However, the determination of these factors is necessary to study the behaviour of cracking.

The eigenvalues and their corresponding eigenvectors can be obtained for each case by solving equation (2), and the stress singularities may be determined. The whole asymptotic fields near the crack-tip can be found by equation (3) according to the form of eigenvalues. After calculating the relative displacements of the crack lips near the crack-tip by finite element method, the factors k_1 and k_2 are calculated by using kinetic method. The following examples are the application of the developed model for the different cases discussed above.

Example 1: crack perpendicular to the interface ($\theta_0 = 90°$)

In this particular case, Cook and Erdogan (1) have proposed an analytical solution for a crack terminating perpendicularly to the interface of two infinite media with an uniform pressure on the crack lips. It is the case of pure mode I. In order to verify our developed model, the stress intensity factors K_I for different ratios μ_1/μ_2 are calculated and then compared with the analytical solution.

Let consider a plate (40mm × 40mm) formed by two materials with a central crack (length 2*a*=2mm). The dimension of the plate is so big with respect to the crack that the plate can be considered as infinite one. Young's modulus of material 1 equals to 1 Mpa. The uniform pressure $\sigma_{yy} = 1$ MPa. The results of the comparison are shown in table 1. It can be observed that the results obtained according to our model agree well with the analytical solution. Some small errors for K_I are due to the calculation of finite element method which is employed in order to determine the u_θ displacements on the crack lips. These displacements are used to obtain k_I by the kinetic method according to equation (3) and then the stress intensity factor K_I.

Example 2: inclined crack with respect to the interface ($\theta_0 = 45°$)

The same plate as before is considered. The inclined angle between the crack and the interface is 45°. An uniform tensile ($\sigma_{yy} = 1$ MPa) is applied on the two extremities in the Y-direction. The calculated values of k_1 and k_2 are listed in table 2. In order to compare the developed displacement fields with the results obtained by finite element, the displacements of the points on a cercle around the crack-tip with r = 0.1a are calculated. Figure 4 shows one of the comparison results between these two displacement fields. The displacement fields obtained by the developed model agree well with the r finite element results.

$\mu 1/\mu 2$	λ	K_I (MPa*(mm)$^{1-\lambda}$) (reference 1)	(present paper)
0.00720	.73345	4.922	4.968
0.04330	.71103	4.176	4.190
1.0000	.5	1	0.960
23.080	.17575	0.074	0.068
138.46	.07491	0.0079	0.010

TABLE 1 - K_I for a crack perpendicular to the interface (θ_0=90°)

$\mu 1/\mu 2$	k_1 (MPa*(mm)$^{1-\lambda_1}$)	k_2 (MPa*(mm)$^{1-\lambda_2}$)
0.0072	-8.8760	9.1464
0.0433	-4.0710	4.6129
0.5000	-1.3911	-0.9857
2.0000	0.8018	-1.5737
23.080	1.1009	6.2239
138.46	1.2705	-8.1590

TABLE 2 - Factors k_1 and k_2 for inclined crack (θ_0=45°)

CRACK BIFURCATION AND DISCUSSIONS

When finding the asymptotic fields and the factors k_1 and k_2, it is possible to predict the bifurcation of an unstable crack. According to the criterion of the maximum circumferential stress $\sigma_{\theta\theta}$, the bifurcation angle will satisfy the following equations:

$$\lim_{r\to 0} \sigma_{r\theta} = 0 \quad \text{and} \quad \lim_{r\to 0} \frac{\partial \sigma_{r\theta}}{\partial \theta} = 0 \tag{4}$$

It results in:

$$k_1 r^{\lambda_1 - 1} \tilde{\sigma}_{r\theta_1}(\theta) + k_2 r^{\lambda 2 - 1} \tilde{\sigma}_{r\theta_2}(\theta) = 0$$

and

$$k_1 r^{\lambda_1 - 1} \frac{\tilde{\sigma}_{r\theta_1}(\theta)}{\partial \theta} + k_2 r^{\lambda 2 - 1} \frac{\tilde{\sigma}_{r\theta_2}(\theta)}{\partial \theta} < 0 \tag{5}$$

When $\theta_0 = 90°$ or $\theta_0 = \theta t$, a single real value is found, $\lambda=\lambda_1=\lambda_2$, the asymptotic fields are similar to those for homogeneous material.

When $\theta_t<\theta_0<90°$, the case of a pair of real eigenvalues, equation (5) can be written as:

$$r^{\lambda_1 - 1}[k_1 \tilde{\sigma}_{r\theta 1}(\theta) + k_2\, r^{\lambda_2 - \lambda_1}\, \tilde{\sigma}_{r\theta 2}(\theta)] = 0 \qquad (6)$$

When $r \to 0$, the left part of equation (6) tends to the infinite. However, because of $\lambda_2 > \lambda_1$ so, $r^{\lambda_2 - \lambda_1} \to 0$ when $r \to 0$. It means that the bifurcation direction will be governed only by the first term of equations (5), i.e. by the smaller eigenvalue λ_1. The other eigenvalue λ_2 does not influence the bifurcation direction. Only two propagation directions are possible according to the sign of k_1 to ensure $k_1 \dfrac{\partial \tilde{\sigma}_{r\theta 1}}{\partial \theta} < 0$. The values of the bifurcation angles for different ratios μ_1 / μ_2 and different crack incidence angles θ_0 are given in table 4.

When $0° \le \theta_0 < \theta_t$, the stress singularities take oscillatory nature. The bifurcation criterion can not be used directly. Another work has to be developed.

θ_0 / μ_1/μ_2	80	70	60	50	40	30	20	10
0.2	-94.54 50,15	-98.67 52.72	-98.53 56.57	-91.24 58.24	-76.99 62.25			
0.3	-93.88 49.91	-97.91 50.43	-99.50 51.99	-96.78 53.84	-88.20 56.19			
0.5	-94.21 49.31	-98.27 48.32	-100.94 48.10	-101.05 49.07	-96.97 50.39	-86.02 55.9		
2	89.70 -55.91	85.45 -59.51	81.37 -63.23	77.41 -66.9	73.50 -70.62	69.38 -74.53	63.94 -79.74	
3	90.50 -56.39	85.90 -60.35	81.52 -64.34	77.30 -68.33	73.19 -72.31	69.02 -76.36	64.16 -80.93	
5	91.80 -56.2	86.85 -60.45	82.18 -64.74	77.72 -69.04	73.43 -73.33	69.17 -77:62	64.63 -82.01	57.16 -88.91

TABLE 3 - Two possible bifurcation angles for different μ_1/μ_2 ratios and different values of angles θ_0

CONCLUSION

This work shows how to take into account bimaterial stress singularity at a crack tip. This allows to determine : a) a characteristic angle θ_t (function of μ_1 / μ_2) which governs the form of the singularity ; b) the asymptotic displacement and stress fields ; c) the bifurcation angle of the crack.

REFERENCES

(1) Cook, T.S. and Erdogan, F., Int. J. of Eng.Science, No.10, 1972, pp. 677-697

(2) Fenner, D.N., Int. J. of Fracture, Vol.12, No.5, 1976, pp. 705-718

(3) Williams, M.L., Bull.of the seismological soc.of America, Vol.49, No.2, 1959, pp.199-204

(4)Li, J., Zhang X.B. and Recho, N, Int. J. of Eng. Fracture Mech., *to be published*

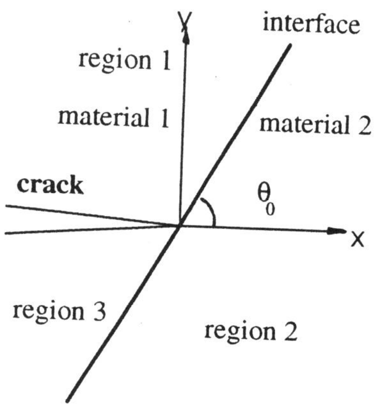

Fig. 1. Crack terminating at the interface of two elastic materials

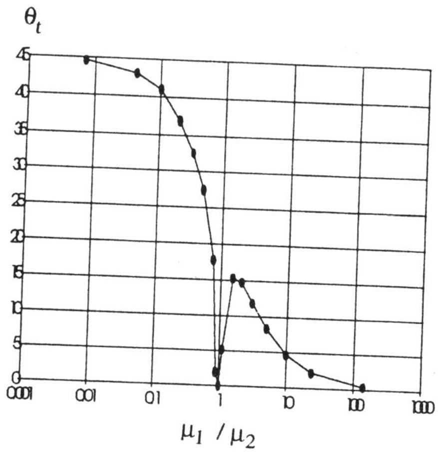

Fig. 2. real-complex eigenvalue transition angles θ_t

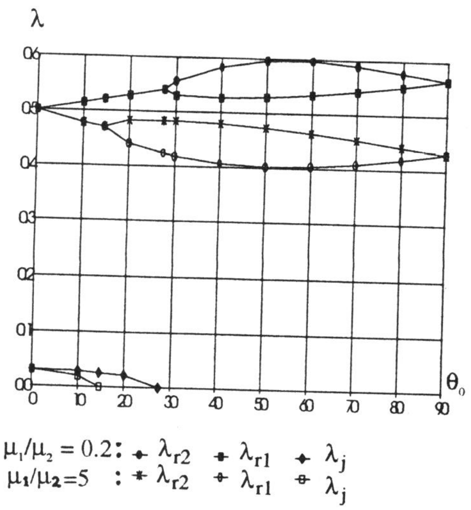

Fig. 3. Variation of eigenvalues against incidence angle θ_0

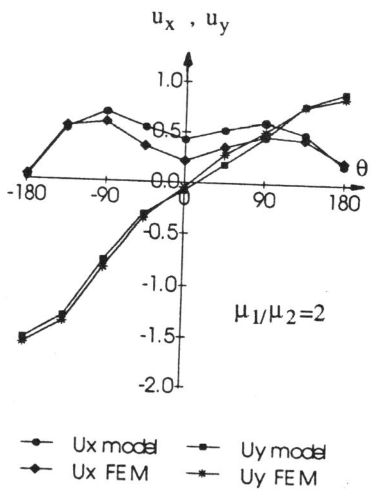

Fig. 4 Comparison between the two displacement fields