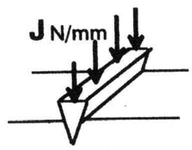
SIMPLE FORMULA FOR STRESS NEAR A CRACK TIP IN A HARDENING MATERIAL

R. L. Roche*

Simple and approximate formula giving stress as a function of the polar coordinates (radius and angle) for the HRR singularity are proposed. The deformation theory of plasticity with a material power law is considered (incompressible material). Only plane strain in mode I and proportional loading (no unloading, no initial nor thermal stresses,...) is studied. A few examples of application are given. Current practice in writing HRR singularity is discussed. An appendix give indications on the way the principle of complementary work is used for getting formula and how to apprise their accuracy.

INTRODUCTION

The singular behavior near a crack tip in a hardening material (the "HRR singularity") was studied by Hutchinson [1] and Rice [2], and is often described in books on fracture [3]. No simple analytical expressions giving stress as a function of the polar angle was found. The aim of this paper is proposing approximate ones, simple for easy use and accurate enough for reliable results. It will be proposed here the same ones as in a preceding publication [3] (with cosmetic improvements), more attention being given to the way followed for writing them (in the appendix).



A power law hardening material is considered here. The material constitutive equation is "finite plasticity" (non linear elasticity) $\varepsilon^*/\varepsilon_0=(\sigma^*/\sigma_0)^n$ where $\sigma^*=(1.5s_{ij}s_{ij})^{0.5}$, $\varepsilon^*=(2\varepsilon_{ij}\varepsilon_{ij}/3)^{0.5}$ are the Von Mises equivalent stress and strain, $\varepsilon_{ij}=2s_{ij}\varepsilon^*/3\sigma^*$ the strain components (incompressible material) and $s_{ij}=\sigma_{ij}-\delta_{ij}\sigma_{kk}/3$ the deviatoric part of the stress tensor σ_{ij}. The loading is proportional in mode I. It is not given by conventional forces, but by the material one J [4], [5] such the work $J\delta a$ is needed for a growth δa of the crack length (thickness equal to one). This material force can be illustrated as pushing on a fictitious knife on the crack tip. Here it is easy to see that this material force is equal to the well known J integral [9](see appendix).

* Roche Consult. S.I.D.R., Meudon, France

THE WAY FOR GETTING APPROXIMATE FORMULA

Satisfying the equations of equilibrium is essential, the best way to meet this condition is deriving stress components from the Airy stress function. The current practice[1] is choosing $r^{(2n+1)/(n+1)}.f(\theta)$ and so is done her, allowing to write the stress components:

$\sigma_{rr} = [(2n+1)/(n+1)].f + f''$ $\quad \sigma_{\theta\theta} = [n(2n+1)/(n+1)^2].f$ $\quad \sigma_{r\theta} = -[n/(n+1)].f'$

$f(\theta)$ depending on m parameters p_i, its choice is subjective, nevertheless some conditions are imposed by equilibrium (traction free crack surface): $f(\pi) = f'(\pi) = 0$ and by mode I (symmetry): $f'(0) = f'''(0) = 0$. It could be power series of $\cos(\theta/2)$, but Fourier series are more convenient: $f(\theta) = \Sigma_{i=0}^{m} p_i.\cos(i\theta/2)$. This is the choice made here, with few parameters for getting formula short enough (see appendix).

With the help of the *principle of complementary work* the parameters p_i are determined as functions of the hardening exponent n. What is needed for simple formula are analytical expressions, then some ones must be taken to represent the computed numerical values.

RESULTS AND APPLICATIONS

$$S = \sigma_o.\left[\frac{J}{(1+n^{-1/3}).\pi.\sigma_0\varepsilon_0.r}\right]^{1/(n+1)} \qquad S = \text{maxi of } \sigma^* = \frac{\sqrt{3}}{2}\sqrt{(\sigma_{rr}-\sigma_{\theta\theta})^2 + 4\sigma_{r\theta}^2}$$

$$\sigma_{rr} = S\frac{2n(n+2\sqrt{3}-1)}{3(n+1)^2(n+4)}\left[\frac{5(7n+3)}{4n}\cos(\tfrac{\theta}{2}) + (n-1)\cos(\tfrac{2\theta}{2}) - \frac{5(n+5)}{12n}\cos(\tfrac{3\theta}{2}) - \frac{(n-1)(7n+8)}{21n}\cos(\tfrac{6\theta}{2}) + \frac{22(2n+1)(n-1)}{21n}\right]$$

$$\sigma_{\theta\theta} = S\frac{2n(n+2\sqrt{3}-1)(2n+1)}{3(n+1)^3(n+4)}\left[5\cos(\tfrac{\theta}{2}) + (n-1)\cos(\tfrac{2\theta}{2}) + \tfrac{5}{3}\cos(\tfrac{3\theta}{2}) + \tfrac{n-1}{21}\cos(\tfrac{6\theta}{2}) + \frac{22(n-1)}{21}\right]$$

$$\sigma_{r\theta} = S\frac{2n(n+2\sqrt{3}-1)}{3(n+1)^2(n+4)}\left[\tfrac{5}{2}\sin(\tfrac{\theta}{2}) + (n-1)\sin(\tfrac{2\theta}{2}) + \tfrac{5}{2}\sin(\tfrac{3\theta}{2}) + \tfrac{n-1}{7}\sin(\tfrac{6\theta}{2})\right]$$

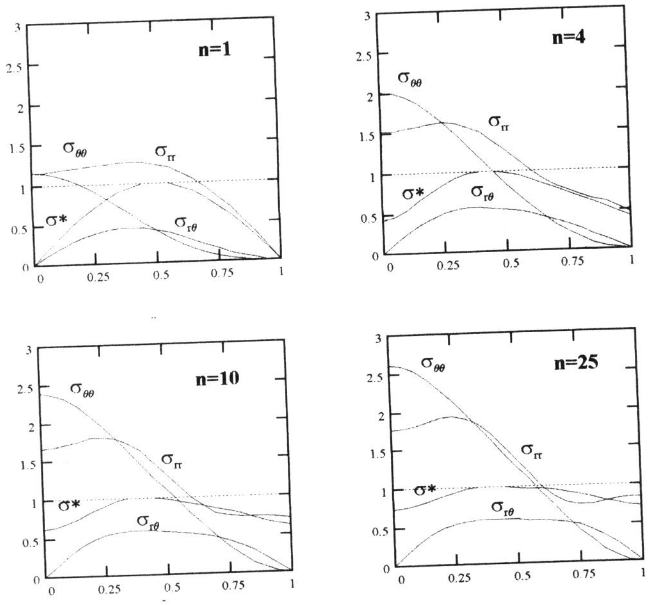
---

1 There is not enough place to discuss this point, see [1] and [2]. Nevertheless it is useful to point out that some conditions are required. *First:* if all strain components are multiplied by a factor ξ, all the stress ones are multiplied by one factor $\Sigma(\xi)$ depending only on ξ (this condition is met by a power law, but not by the Ramberg Osgood law for the effective Poisson ratio is depending on the strain) *Second:* HRR singularity is expending all over the plane (practically a large extension in regard of blunting) *Consequence:* the radius r has only a scaling effect on all mechanical quantities, for instance all the displacements components can be written R(r).function(θ), displacement gradient components and strain ones are proportional to R', stress components to Σ(R') and J integral (on a circle) to R'.Σ(R').r. As J is no path dependent (see appendix), R'.Σ(R') is varying as 1/r. In other words, *along any radius any product of any strain component by any stress one (at the same point) is varying as 1/r.*

They are illustrated on figures giving stress as a function of the angle θ, from zero at the opposite of the crack to π on the crack (for n=1, 4, 10, 25) [S=1].

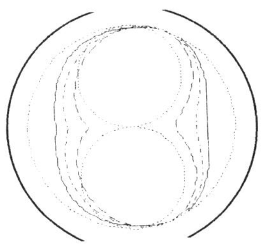
When n=1 formula are those of the linear elastic material non compressible (Poisson'ratio = 0.5).

For large n , formula are near those given by the slip-line field.

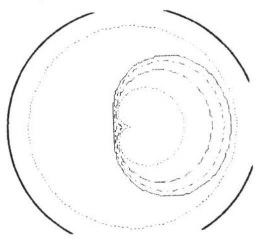


They allow studies on the behavior near the crack tip. The figure on the other side show some examples. diagrams are presented for n=1,4, 10,25. and maximal equivalent stress S = 1. .

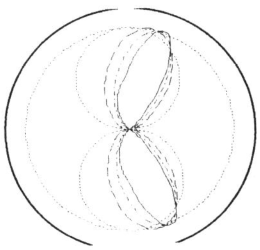
Crack shape and **CTOD** can also be analyzed.



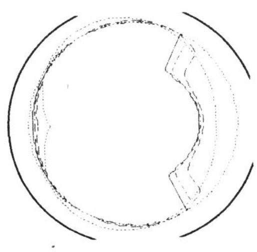
Equivalent stress



$\sqrt{3}/(1+\pi)$ hydrostatic tension



Equivalent strain



Displacement

DISCUSSION

Limits of validity It is obvious that approximate formula are not fully accurate, the proposed ones give stresses with an error not exceeding few percents (for $n<25$), strain and displacement values are not so good. It must be pointed out that most of the limits for application are not due to the simplifications made here. Formula are not valid too near the crack tip, the hypothesis of linearity between strain and displacement being not correct where blunting is significant. They are not valid too far from the crack tip, there are several causes for that. The HRR singularity does not extend all over the plate and the connection to the far stress field must be considered. Plane strain is only valid when the thickness is large in regard to the distance to the crack tip. And for small loading, the HHR singularity could be embedded in a linear one.

Comparison with results available in open literature. It can be seen that stresses given by the proposed formula are near those given in preceding publications. There is a small difference in the expression of the maximal equivalent stress, it is to say of what is noted I_n ; this difference is not troublesome for practical applications (the formula give $I_1 = 2\pi$ for an incompressible linear elastic material (n=1), it is the correct value, the values given in [1] seem going towards another one). The present formula do not use Young modulus, for it is not included in the input, being absent from a power law. It must be pointed out that in [1] the constitutive equation is the Ramberg Osgood one, and the loading is not the material driving force J but an linear elastic tension field σ_∞ , as a mater of fact computations were made on a power law and J (see equation (23)), then they were translated in assuming hypothesis (small scale yielding,...).

Today writing of HRR singularity Such a way was fully justified when J was only estimated for linear materials, but now it is possible to compute J in plastic regions (there is an excellent handbook [7]) and to base the description of HRR singularity on J. The today practice is yet keeping too much of the old way, like Young modulus and Poisson ratio (what value, 0,5 or elastic one?) and assuming that the HRR singularity is always embedded in a linear one. As most of the cracks appear in plastic regions (stress concentrations) there is no justification at all to admit that a linear singularity connect the HRR singularity to the far field. The right way should be the straight computation of J in the plastic stress field and avoiding useless assumptions like small scale yielding.

CONCLUDING REMARKS

The proposed formula are simple, they can be used for manual computations, or on a pocket computer, the best way being using a spread sheet. Their main advantage is giving stress as a function of the polar angle.

They are accurate enough (error less than few percent if the hardening exponent does not exceed 25) for practical applications (hydrostatic tension, strain, displacement, CTOD,...).

As most of the cracks appear in plastic regions it is useless to say the HRR singularity is embedded in a linear one and to introduce quantities like Young modulus, Poisson' ratio.

APPENDIX: THE USE OF THE PRINCIPLE OF COMPLEMENTARY WORK

Of all the boundary force T_i and stress σ_{ij} fields that satisfy the equations of equilibrium the "actual" one is such that its (small) variations are fulfilling $\iint \varepsilon_{ij}.\delta\sigma_{ij}.ds = \int u_i \delta T_i$ ε_{ij} and u_i being corresponding strain and displacement. (in non linear elasticity $\delta W_c = \int u_i \delta T_i$ where $W_c = \iint (\varepsilon^* . \sigma^*)/(n+1)].ds$). In other words this principle is equivalent to the equations of compatibility if *all* the fields satisfying the equations of equilibrium are considered. Such a condition is not convenient for applications and a trick is used in practice: only a part of all these stress fields is considered, each of them being defined by a set of m parameters p, and this set (this vector) is obtained from the m equations $W_{c,p} = \int u_i . T_{i,p}$. [writing $F(p,...)_{,p} = \partial F/\partial p$ and $F(x_1, x_2, ...)_{,i} = \partial F/)x_i$]. For having stress fields satisfying the equation of equilibrium , a good way is taking Airy stress functions depending on m parameters p_i. It is easy to deduce stress and strain components σ_{rr}, $\sigma_{\theta\theta}$, $\sigma_{r\theta}$, $\varepsilon_{rr} = -\varepsilon_{\theta\theta}$, $\varepsilon_{r\theta}$ applied forces $T_r = \sigma_{rr}$, $T_\theta = \sigma_{r\theta}$, complementary energy W_c from them. Unfortunately, displacements are not known, otherwise the correct solution would be. At first glance, displacement can be deduced from strain, but that cannot be rigorously made. Compatibility equations are not fully meet and there are three strain components, three equations between these components and the two ones of displacement which is not possible to meet together.

An other way must be taken. Equivalence between the principle of complementary work and equation of compatibility was contested, a check of it was given by Southwell [7]. The basic equation[2] can be changed in $\iint L.\delta\phi = 0$ where ϕ is any Airy function and L the Lamé function (L=0 is the equation of compatibility written for the first time in 1860 by Barré de Saint Venant). It's obvious that when ϕ is any function L=0 and equation of compatibility is met, but for practical application this only show that an average of L weighed by ϕ variations should be convenient. This is the way followed here, the m unknowns parameters p are given by the m equations $\iint L.\phi_{,p} = 0$. In polar coordinates

$$L = \varepsilon_{rr,\theta\theta} - r.\varepsilon_{rr,r} + r^2.\varepsilon_{\theta\theta,rr} + 2r.\varepsilon_{\theta\theta,r} - 2r.\varepsilon_{r\theta,r\theta} - 2.\varepsilon_{r\theta,\theta}$$

and here

$$L = r^{n/(n+1)}.\{\varepsilon''_{rr} + [(n.(n+2)/(n+1)^2].\varepsilon_{rr} - [2/(n+1)].\varepsilon_{r\theta}\}$$

(ε' derivative of ε related to θ).

The conventional writing of the principle can be used for knowing the displacement:

$u_r = (n+1)r\varepsilon_{rr}$ and $u'_\theta = -(n+2))r\varepsilon_{rr}$, it is to say $u_\theta = -(n+2))r.\int_0^\theta \varepsilon_{rr}.d\theta$.

Such a result deserve some comments. It is easy to see that the two equations giving ε_{rr} and $\varepsilon_{\theta\theta}$ are satisfied, but no the one giving $\varepsilon_{r\theta}$. The discrepancy between the value of $\varepsilon_{r\theta}$ related to the stress field and the one related to the displacement give some idea of the accuracy of the formula, this difference is $\varepsilon_{r\theta} - \{[(n+2)/2].\varepsilon'_{rr} + [n(n+2)/2(n+1)].\int_0^\theta \varepsilon_{rr}.d\theta\}$. Obviously it is the equation of compatibility

---

[2] In three dimensional cases, it must be written $\iiint L_{ij}.\delta\phi_{ij}.dv = 0$ where ϕ_{ij} is a symmetrical tensor giving a stress field satisfying the equations of equilibrium $\sigma_{il} = e_{ijk}.e_{lmn}.\phi_{jm,kn}$ (the functions of J. C Maxwell and of G. Moreira) where e_{ijk} is the permutation tensor (fully antisymetrical) and L a symmetrical tensor, the components of it $L_{il} = e_{ijk}.e_{lmn}.\varepsilon_{jm,kn}$ corresponding to those of the Riemann-Christoffel curvature tensor $R_{jm.kn}$ and therefore meeting the Bianchi'identities $L_{ij,j} = 0$. Obviously $L_{ij} = 0$ are the six equations of compatibility.

The equations are not able to give the value of the parameters, but only them multiplied by an unknown factor (they give their ratios). The reason for that is the loading intensity has not be introduced (T is proportional to σ). As indicated at the beginning this loading is defined by material forces. The present case is simple (plane crack, homogenous and isotropic material, no initial or thermal stress, no unloading) and the material force is only a vector J parallel to the x-axis. The field is invariant in a translation along this x-axis, hence the theorem of Emi Noether is applicable, there is an constant integral which is the path one proposed by Rice [9] and is the material force loading the plate. In polar coordinates: $J=\int\{w.\cos\theta - \sigma_{rr}[u_{r,r}\cos\theta - (u_{r,\theta} - u_\theta)\sin\theta/r] - \sigma_{r\theta}[u_{\theta,r}\cos\theta - (u_{\theta,\theta} + u_r)\sin\theta/r]\}.ds$ w being the energy density $n\sigma^{*n+1}/(n+1)$. Here u is proportional to $r^{1/(n+1)}$ and the path is a circle (two half a circle), hence $J=2\int_0^\pi\{[n\sigma^{*n+1}/(n+1)\cos\theta] - \sigma_{rr}[u_r\cos\theta/(n-1) - (u_{r,\theta} - u_\theta)\sin\theta/r] \ldots$

$\ldots -\sigma_{r\theta}[u_\theta\cos\theta/(n-1) - (u_{\theta,\theta} + u_r)\sin\theta/r]\}.rd\theta$

the computation ot it giving the last unknown as a function of J. All the parameters values are now known as a function of the hardening exponent n. The last task is choosing simple analytical expressions to introduce them in the formula.

REFERENCES

[1] Hutchinson, J.W.	1968	J. Mech. Phys. Solids, Vol 16, pp.13 to 31
[2] Rice, J. R	1968	J. Mech. Phys. Solids, Vol 16, pp.1 to 12
[3) Anderson, T. L.	1991	"Fracture Mechanics", CRC Press, Boca Raton, Fl.
[4] Roche, R L	1995	Revue Française de Mécanique, n°195-4, pp. 299 to 307
[5] Roche, R L	1979	Int. J. Pres. Ves. & Piping, Vol. 7, pp 65 to 82
[6] Roche, R L	1981	in "Adv. in fracture mechanics" Pergamon, pp. 97 to 105
[7] Kumar, V.	1981	EPRI Report NP 1931 (Res. Proj. 1237-1)
[8] Southwell R. V.	1936	"Theory of elasticity" Clarendon Press. Oxford
[9] Rice, J. R	1968	J. Appl. Mech., june 1968, pp. 379 to 385