ON THE USE OF DIMENSIONAL ANALYSIS IN FRACTURE MECHANICS

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This work presents an overview on the use of Dimensional Analysis in Fracture Mechanics. Applying the Dimensional Analysis methodology shows that the classic base used in Newtonian Mechanics {L, M, T} is also valid for Fracture Mechanics. A "partially discriminated" base $\{L_x, L_y = L_z, M, T\}$ can be used to obtain more accurate solutions. Following the presented methodology it is possible to approach any problem related to a material's failure. As an example, the determination of the stress intensity factor for an edge-cracked plate in bending is developed in detail.

INTRODUCTION

Dimensional Analysis dates back to 1888 when Fourier introduced the concept of dimension in his book "Théorie Analytique de la Chaleur" (1). At the end of the 19th. century, Lord Rayleigh applied Dimensional Analysis to solve problems which had insuperable mathematical difficulties (2). In 1914, Buckingham provided a practical rule to determine the number of dimensionless quantities that are relevant in a particular physical problem (3). The complete development of the Dimensional Analysis theory is due to the works of Bridgman (4) and Palacios (5).

Nowadays, this method can be applied when the complete mathematical formulation of a physical problem has not been given, helping in the investigation of the nature of the solution. Applying this method to a problem the number of inherent variables is reduced, thus proving to be an invaluable tool for experimenters. Also of great interest in design are the nondimensional coefficients and parameters which can be obtained from this type of analysis. The main advantage of the method is that it is both simple and rapid. Its principal shortcoming, however, is that it does not provide such complete information as might be obtained by carrying out (if possible) a detailed analysis. The solutions obtained from Dimensional Analysis are always composed of an undetermined function of dimensionless quantities.

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Experimental work must be used in order to determine these unknown functions. Therefore, Dimensional Analysis has to be considered as an auxiliary tool, particularly useful when analytical solutions cannot be obtained by classic methods.

There are some fields in which Dimensional Analysis has had virtually no application. To the best of our knowledge, Fracture Mechanics is one of such fields. Attempts have been made by Wagner on the problem of the failure of materials (6) and by Navarro and De los Ríos (7) in fatigue (7). Even Rice et al. used Dimensional Analysis as an auxiliary tool in determining J directly from the load-displacement curve of a single specimen (8), but a completely rigorous work has not been developed until now (9). Nevertheless, extensive work exists on related topics such as size effects (10, 11) or model laws (12) but these are outside the scope of this paper.

DIMENSIONAL ANALYSIS AND FRACTURE MECHANICS

Inside the framework of Dimensional Analysis theory, Linear Elastic Fracture Mechanics (LEFM) can be formulated using the following set of fundamental equations:

$$F = m_i \frac{d^2 s}{dt^2} \quad [1] \qquad F = \frac{m_g \cdot m_g'}{d^2} \quad [2]$$

$$m_g = \sqrt{G_u} \cdot m_i \quad [3] \qquad \sigma = E \cdot \varepsilon \quad [4]$$

$$\tau = \mu \cdot \gamma \quad [5] \qquad dW_s = R \cdot dA \quad [6]$$

where [1], [2] and [3] are a complete set of fundamental equations in Newtonian Mechanics, [4] and [5] represent the most simplified Hooke's Law expression and the last one [6] is a cracking condition in the material (energy per unit area needed to create new surfaces or cracking resistance, R).

The matrix of dimensional coefficients for this set of equations can be expressed in the form:

	[s]	[m_i]	[t]	[F]	[m_g]	[G_u]	[E]	[μ]	[R]
[1]	-1	-1	2	1	0	0	0	0	0
[2]	2	0	0	1	-2	0	0	0	0
[3]	0	-1	0	0	1	-1/2	0	0	0
[4]	-2	0	0	1	0	0	-1	0	0
[5]	-2	0	0	1	0	0	0	-1	0
[6]	-1	0	0	1	0	0	0	0	-1

Its rank is h = 6 and therefore a base must be integrated by p = n - h = 9 - 6 = 3 variables. The minor generated by the columns [F], $[m_g]$, $[G_u]$, [E], [μ] and [R] is different from zero and the dimensional base can be constitued by the remaining independent quantities: {[s], $[m_i]$, [t]} = {L, M, T}.

Huntley suggested using the "discriminated base" $\{L_x, L_y = L_z, M, T\}$ in order to obtain more accurate solutions from the Dimensional Analysis methodology (13). However, the stress state in the vicinity of a crack tip of a body is commonly expressed as a function of polar coordinates (r, θ) as shown in Figure 1 for mode I configuration. The r coordinate is geometrically defined by

$$r = \sqrt{y^2 + z^2} \qquad [7]$$

Consequently, directions Y and Z are equivalent and cannot be discriminated (9). Therefore in LEFM it is possible to use only the "partially discriminated" base $\{L_x, L_y = L_z, M, T\}$ as shall be shown in the worked example.

METHODOLOGY

In order to apply the Dimensional Analysis methodology to a particular problem it is convenient to follow the next steps:

- Consider all the relevant variables $(x_1, \ldots, x_{n-1})$ involved in the problem. The most general solution can be expressed as an implicit function:

$$F(x_1, x_2, \ldots, x_{n-1}; y) = 0 \qquad [8]$$

"y" being the variable that one wishes to express explicitly as a function of the rest.

- Make the matrix of dimensional coefficients corresponding to all these variables using the {L, M, T} base or $\{L_x, L_y = L_z, M, T\}$ if possible.

- Calculate the rank (h) of the matrix. A complete set of dimensionless quantities is composed of r = n - h independent nondimensional products.

- Select h independent variables and calculate the r nondimensional products (π_i) by equating the corresponding dimensional coefficients:

$$\pi_i = x_1^{\varepsilon_1} x_2^{\varepsilon_2} \ldots x_h^{\varepsilon_h} x_i^{1} \quad ; \quad h+1 \le i \le n \qquad [9]$$

- In dimensionless variables the solution of the problem can be expressed as:

$$\Psi(\pi_1, \ldots, \pi_r) = 0 \qquad [10]$$

- Making use of the Theory of Generalized Homogeneous Functions it is possible to find the value of "y" as a function of the remaining nondimensional products.

Note that the number of variables in the solution has been reduced from n to r = n-h. There are some criteria for selecting the h independent variables:

a) The variable "y" cannot be selected in order to find out its value at the end of the process.

b) If possible, the "cause" must not be selected so that Hooke's postulate: "effect" is proportional to "cause" may be used.

c) Any variable possessing proportionality properties with "y" has not to be selected so these properties can be used as a simplification in the final general equation.

In order to clarify the proposed method a worked example is presented below.

APPLICATION

This worked example deals with the determination of the stress intensity factor for an edge-cracked plate in bending, as shown in Figure 2.

Following this arrangement the relevant variables on the problem are:

a) Plate geometry: thickness (B), width (W),
b) Crack geometry: length (a),
c) Stress state: load (P), span (L),

K_I is the variable to be expressed as a function of the rest. Thus, the solution can be expressed in the form:

$$F(B, W, a, L, P; K_I) = 0 \quad [11]$$

Using the base {L, M, T} the matrix of dimensional coefficients is:

	B	(W)	a	L	(P)	K_I
L	1	1	1	1	1	-1/2
M	0	0	0	0	1	1
T	0	0	0	0	-2	-2

Its rank is h = 2 and W and P are selected as independent variables. The number of nondimensional products is r = n - h = 6 - 2 = 4. Therefore, the following geometrical factors may be defined:

$$\overline{\omega}_B = \frac{B}{W} \quad ; \quad \overline{\omega}_L = \frac{L}{W} \quad ; \quad \overline{\omega}_a = \frac{a}{W}$$

and also by equating dimensional coefficients:

$$\left[\pi_{K_I}\right] = \left[W^{\varepsilon_W} P^{\varepsilon_P} K_I\right] = L^0 M^0 T^0 \quad \Rightarrow \quad \pi_{K_I} = \frac{K_I W^{3/2}}{P}$$

Expressing [11] as a function of nondimensional variables and making use of the Theory of Generalized Homogeneous Functions:

$$K_I = \frac{P}{W^{3/2}} f\left(\frac{B}{W}, \frac{L}{W}, \frac{a}{W}\right) \qquad [12]$$

But, if one considers the "partially discriminated" base $\{L_x, L_y = L_z, M, T\}$ the matrix can be expressed as:

	(B)	(W)	a	L	(P)	K_I
L_x	1	0	0	0	0	-1
$L_y=L_z$	0	1	1	1	1	1/2
M	0	0	0	0	1	1
T	0	0	0	0	-2	-2

In this case h = 3 and a base for the problem could be {B, W, P}. The number of dimensionless products is now r = n - h = 6 - 3 = 3 (one less than above). Thus, it is possible to define:

$$\overline{\omega}_L = \frac{L}{W} \quad ; \quad \overline{\omega}_a = \frac{a}{W}$$

and also

$$\left[\pi_{K_I}\right] = \left[B^{\varepsilon_B} W^{\varepsilon_W} P^{\varepsilon_P} K_I\right] = L_x{}^0 L_{y\equiv z}{}^0 M^0 T^0 \quad \Rightarrow \quad \pi_{K_I} = \frac{K_I B \sqrt{W}}{P}$$

Consequently, the solution can be explicitly expressed as:

$$K_I = \frac{P}{B\sqrt{W}} f\left(\frac{a}{W}, \frac{L}{W}\right) \qquad [13]$$

that is the most complete solution obtainable from the Dimensional Analysis methodology. However, having basic knowledge in Strength of Materials, the bending moment for this particular test configuration is

$$M = \frac{P \cdot L}{4} \qquad [14]$$

M is proportional to σ and the latter has the same property with K_I. Therefore, $K_I \propto L$ and consequently the solution can be expressed as

$$K_I = \frac{P\,L}{B\,W^{3/2}} f'\left(\frac{a}{W}\right) \qquad [15]$$

The solution usually found in bibliography (14) is:

$$K_I = \frac{PL}{BW^{3/2}}\left[2.9\left(\frac{a}{W}\right)^{1/2} - 4.6\left(\frac{a}{W}\right)^{3/2} + 21.8\left(\frac{a}{W}\right)^{5/2} - \ldots\right] \qquad [16]$$

that is identical in form to [15] if

$$f'\left(\frac{a}{W}\right) = \left[2.9\left(\frac{a}{W}\right)^{1/2} - 4.6\left(\frac{a}{W}\right)^{3/2} + 21.8\left(\frac{a}{W}\right)^{5/2} - \ldots\right] \qquad [17]$$

This undetermined function f'(a/W) could readily be determined from experimental work. Note that the number of variables has been reduced from six in [11] to one in the undetermined function of the final solution [17].

CONCLUSIONS

The following concluding remarks can be obtained from the present work:

1. Dimensional Analysis methods have been proved to be simple and rapid in determining the solution of any physical problem, especially when analytical solutions are not available.

2. Since the obtained solution is always composed of an undetermined function, Dimensional Analysis should be considered only as a useful auxiliary tool for experimenters.

3. The main advantage of the method is the significant reduction in the number of variables involved in the problem by considering nondimensional parameters. Nevertheless a deep knowledge of the physical problem is required to select the correct variables involved.

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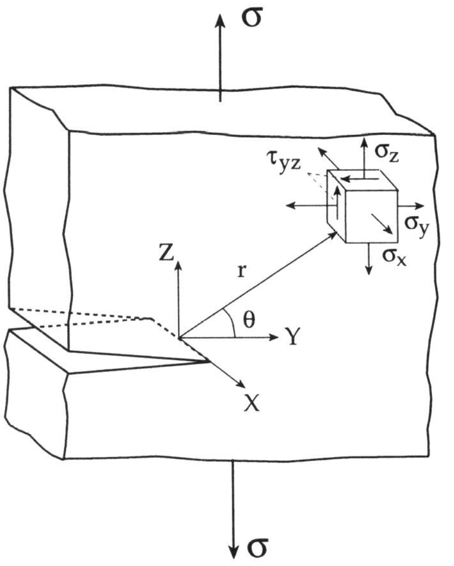

$$\sigma_{ij} = \frac{K_I}{\sqrt{2\pi r}} f_{ij}^{I}(\theta)$$

Figure 1. Stress acting in the vicinity of a crack tip in a solid subjected to a mode I loading.

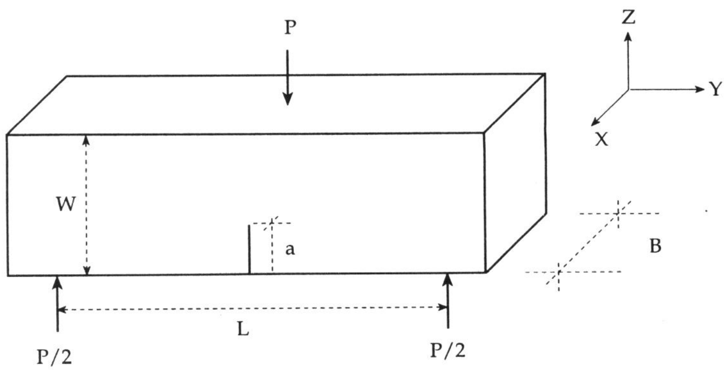

Figure 2. Sketch of a three point bending test on an edge-cracked plate.