ON A FUNCTIONAL DESCRIPTION OF NON-LOCAL STRENGTH AND FRACTURE. EXISTENCE AND UNIQUENESS.

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A functional approach to non-local strength conditions of continuous and discrete fracture mechanics is extended in this contribution. A functional safety factor and an functional (over)load factor are defined for an analyzed point (or for a potential fracture quantum at discrete fracture). Strength conditions for the point and for the fracture quantum are given. The possibility of a representation of any non-local strength condition in the given form and the representation uniqueness is proved.

INTRODUCTION AND MOTIVATION

In the traditional (local) approach, strength of a body in an analyzed point y is characterized by the value of some function of stress tensor components at the same point without consideration of the stress state in neighbouring points. The local strength condition can be represented e.g. in the form

$$f(\sigma_{ij}(y)) < \sigma_c,$$

where f is a material function and σ_c is a material constant. It gives a good description of experimental data when macro-stress variations are small enough on dimensions of the order of the material structure scale.

There are several problems of strength and fracture mechanics that can not be be solved (or it is tedious to solve) by use of traditional strength conditions. Such problems include the strength small-scale effects, strength description of bodies with singular stress concentrators (corner points, intersection of interfaces) generating singularities with different exponents, the problem of unification of strength conditions for bodies with smooth and singular concen-

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trators, and so on. Some examples of such the problems are given in Fig.1-3.

An example of the strength small-scale effect is presented in Fig.1. An infinite elastic plate with a circle hole is considered that is loaded at the infinity by a uniform traction q. It is known from the elasticity theory that the maximum stress is independent of the hole radius r, is equal to $3q$, and the maximum is realized in the boundary point y. The plate strength evaluated by use of the strength condition $\sigma_{\theta\theta} < \sigma_c$ is then independent of the hole radius and is equal to one third of the strength q_c of the plate without hole. However appropriate fracture tests date (pointed out schematically in the figure) for plates with small holes show that the plate strength depends on the hole radius.

Another example of the strength small-scale effect delivers the same plate but now with a crack instead of a hole, Fig. 2. From linear elasticity one obtain the value for the stress intensity factor $K_1(y) = q\sqrt{\pi l}$. From the linear fracture mechanics one have the strength condition $K_1(y) < K_{1c}$. Using these two expressions, we get the theoretical dependence of the plate strength from the crack length (the solid line), according to which the plate strength tends to infinity as the crack length tends to zero. But the experiments for short cracks show that the strength value tends to a finite value.

The same plate but now with a diamond-shaped hole is considered in the third example (Fig. 3). According to the linear elasticity we have the following stress behavior near the corner points: $\sigma_{ij}(\rho,\theta) \sim K_1(y;\alpha)\rho^{-\gamma(\alpha)} f_{ij}(\theta;\alpha)$. It is impossible to estimate the strength of bodies with such the stress behaviour neither by the traditional local strength condition nor by the local linear fracture mechanics condition. In principle, one can try to use the strength condition $K_1(y;\gamma) < K_{1c}(\gamma)$, which is analogous to the linear fracture mechanics condition. However, one must determine then the critical strength intensity factor K_{1c} experimentally for each γ, it means, for each angle α, what is rather tedious and expensive. Moreover, the same problems occur with the small-scale effect as for short cracks or small circular holes.

These three examples show the necessity of a more general strength theory. Such a theory should describe the small-scale effects and be applicable to bodies without cracks, with cracks as well as with other singular concentrators. This conditions meet the non-local strength theories.

FRACTURE QUANTA AND NON-LOCAL STRENGTH CONDITIONS

In papers Mikhailov (1), (2), a functional approach to non-local strength conditions and fracture criteria is presented. We give in this part the basic notions and ideas of this approach with some modifications.

<u>Definition 1</u>. A body (material) D will be called discretely fracturable if fracture in it can occur at once on a point set $F(D)$ called fracture quantum. The set of all fracture quanta of a body D will be denoted $\mathbf{F}(D)$. □

In a given body D, each fracturing stress field $\sigma_{ij}(x)$ can have, generally speaking, its own fracture quantum. Fracture quanta can differ in position, orientation, form, and dimension. One and the same body point can belong to different fracture quanta. Micro–cracks of a characteristic dimension, micro–pores of characteristic radius and so on can figure as fracture quanta. If isolated body points are fracture quanta then discrete fracture degenerates into point fracture. A geometrical classification of the fracture quanta sets and notions of the homogeneous, isotropic and weakly sensitive to boundary quanta sets are given in (2).

<u>Definition 2</u>. Let a stress field $\sigma(x)$ be given in a discretely fracturable body D possessing a fracture quanta set $\mathbf{F}(D)$. Then for each (potential) quantum $F(\mathbf{F}(D))$ there is a parameter $\lambda'(\sigma; F) > 0$ such that the stress field $\sigma'(x; F) = \lambda'(\sigma; F)\sigma(x)$ causes no fracture of F. The supremum of $\lambda'(\sigma; F)$ for the the field $\sigma(x)$ given and for the quantum F considered will be called the functional safety factor $\underline{\lambda}(\sigma; F)$. If $\lambda(\sigma; F) > 0$, then the stress field $\sigma(x)$ will be called admissible for the quantum F; if $\underline{\lambda}(\sigma; F) = 0$, then inadmissible. We will call the set consisting of all admissible stress fields for a quantum F the admissible stress set $\mathbf{S}(D; F)$. □

In addition to the safety functional $\underline{\lambda}$ we introduce also an (over)load functional $\underline{\Lambda}(\sigma; F) := 1/\underline{\lambda}(\sigma; F)$. It follows from Definition 2 that $\underline{\lambda}$ is a positively-uniform functional of the order -1, $\underline{\Lambda}$ is a positively-uniform functional of the order +1. Both the functionals $\underline{\lambda}$ and $\underline{\Lambda}$ will be called also strength functionals. The strength functionals are characteristics of material and, in general, of body form.

It follows from Definition 2 that the non-local strength condition for a (potential) fracture quantum F can be written in the two equivalent forms:

$$\underline{\lambda}(\sigma; F) > 1, \qquad \underline{\Lambda}(\sigma; F) < 1.$$

The corresponding non-local fracture criterion can be written in the two equivalent forms:

$$\underline{\lambda}(\sigma; F) = 1, \qquad \underline{\Lambda}(\sigma; F) = 1.$$

The global strength condition, i.e. the strength condition for the whole body can be then written in the two equivalent forms:

$$\inf_{F\in\mathbf{F}} \underline{\lambda}(\sigma;F) > 1, \qquad \sup_{F\in\mathbf{F}} \underline{\Lambda}(\sigma;F) < 1.$$

The global (body) admissible stress set is $\mathbf{S}(D) = \cap_{F\in\mathbf{F}}\mathbf{S}(F)$.

The corresponding definitions and non-local strength conditions for the point (not discrete) fracture one can get regarding F as a body point. The infimum, supremum and intersection must be taken over all points of the body D considered. Notions of homogeneous, isotropic weakly sensitive to boundary and finitely non-local quanta strength functionals are given in (2).

Example of non-local strength conditions. We present here, as an example of non-local strength condition of the discrete fracture for a plane body, the strength condition based on average stress over a the fracture quantum (a generalized form of the condition used by Neuber (3), Novozhilov (4) and other authors). The fracture quanta set $\mathbf{F}$ in the condition consists of linear segments having a characteristic length d and being also the domains of non-locality. The strength functional is finitely non-local with internal domains of non-locality, strength homogeneous and isotropic.

$$\underline{\Lambda}_1(\sigma;F) = \max[\underline{\Lambda}_{1+}(\sigma;F),0] < 1, \quad \underline{\Lambda}_{1+}(\sigma;F) := \frac{1}{d_1\sigma_c}\int_F \sigma_{nn}dF.$$

The admissible stress set $\mathbf{S}(F)$ coincides with the space $L_1(F)$ of functions integrable over F. Here σ_c and d are material constants; σ_{nn} is the normal stress component on the segment F. Some other examples are presented in (1), (2).

EXISTENCE AND UNIQUENESS OF THE FUNCTIONAL REPRESENTATION

Existence. Suppose a non-local fracture criterion for a quantum (or a point or a body) is written in the form $\underline{L}(\sigma) = 0$, where $\underline{L}$ is a functional such as $\underline{L}(0) \neq 0$. Let the material considered be such that its fracture is independent of the stress field history. In particular, if a stress field σ is fracturing, then the stress field $\lambda'\sigma$ is also fracturing for any constant $\lambda' \geq 1$. Thus the criterion written means that a stress field σ is not fracturing and causes the strength stable state, when there is $\epsilon > 0$ such that $\underline{L}(\lambda'\sigma) \neq 0$ for all $\lambda' \in [0, 1+\epsilon]$ (we denote the set of such the stresses $\mathbf{S}_s$). On the other hand, a stress field σ causes stable fracture when $\underline{L}(\lambda'\sigma) = 0$ for some $\lambda' \in [0,1)$. The boundary between these two sets is the set of critical or unstable stress fields; a stress field σ belong to this set when $\underline{L}(\lambda'\sigma) \neq 0$ for all $\lambda' \in [0,1)$ and, for any $\epsilon > 0$, there is $\lambda' \in [1, 1+\epsilon]$ such that $\underline{L}(\lambda'\sigma) = 0$.

We should show that there is a positively-uniform functional $\underline{\Lambda}$ of the order +1 such that $\mathbf{S}_s$ coincides with the set of stress fields σ meeting the inequality $\underline{\Lambda}(\sigma) < 1$. Really, using Definition 2 and the fracture criterion in the sense of the previous paragraph, we obtain the safety functional $\underline{\lambda}$ and then the overload functional $\underline{\Lambda}$ having the properties desired.

To get the value of the functional $\underline{\Lambda}(\sigma)$ for any σ practically, it is necessary to calculate the roots $\lambda^* > 0$ of the equation $\underline{L}(\lambda^*\sigma) = 0$ and to assign $\underline{\Lambda}(\sigma) = 1/\inf \lambda^*$ when there are such roots, or to assign $\underline{\Lambda}(\sigma) = 0$ when there are no such roots.

Uniqueness. Suppose, there are two positively-uniform functionals $\underline{\Lambda}_1$ and $\underline{\Lambda}_2$ of the order +1 such that their admissible stress sets $\mathbf{S}_1$ and $\mathbf{S}_2$ coincide and the sets $\mathbf{S}_1^*$ and $\mathbf{S}_2^*$ of stress fields, meeting the corresponding equations

$$\underline{\Lambda}_1(\sigma) = 1, \qquad \underline{\Lambda}_2(\sigma) = 1,$$

coincide too. We will prove that $\underline{\Lambda}_1(\sigma) = \underline{\Lambda}_2(\sigma)$ for all admissible stress fields σ.

Really, let $\sigma \in \mathbf{S}_1$, then $\underline{\Lambda}_1(\sigma) < \infty$. Suppose at first, $\underline{\Lambda}_1(\sigma) \neq 0$. Then $\underline{\Lambda}_1(k\sigma) = k\underline{\Lambda}_1(\sigma)$ for any number $k > 0$. Let $k = 1/\underline{\Lambda}_1(\sigma)$. Then $\underline{\Lambda}_1(k\sigma) = k\underline{\Lambda}_1(\sigma) = 1$, i.e., $k\sigma \in \mathbf{S}_1^* = \mathbf{S}_2^*$ and $\underline{\Lambda}_2(k\sigma) = 1$. It means, $k\underline{\Lambda}_2(\sigma) = 1$ and $\underline{\Lambda}_2(\sigma) = \underline{\Lambda}_1(\sigma)$.

Let now $\underline{\Lambda}_1(\sigma) = 0$. Suppose that $\underline{\Lambda}_2(\sigma) \neq 0$. Repeat then the proof of the previous paragraph interchanging $\underline{\Lambda}_2$ and $\underline{\Lambda}_1$. As a result we get $\underline{\Lambda}_1(\sigma) = \underline{\Lambda}_2(\sigma)$. We obtain the contradiction proving the statement.

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REFERENCES

(1) Mikhailov, S.E., Engng Fract. Mech., Vol.52, 1995, pp. 731-743.

(2) Mikhailov, S.E., Engng Fract. Mech., Vol.52, 1995, pp. 745-754.

(3) Neuber, H., Kerbspannungslehre, Springer, Berlin, 1937.

(4) Novozhilov, V.V., Applied Mathematics and Mechanics (PMM), Vol. 33, 1969, pp. 212-222.

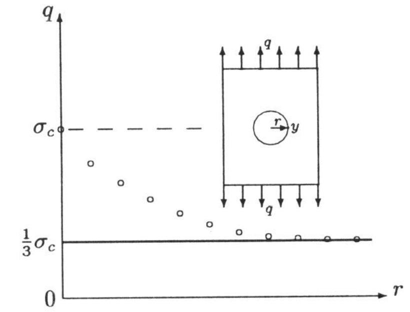


Figure 1. Dependence of the plate strength q on the hole radius r

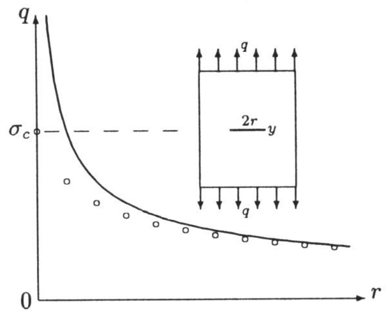
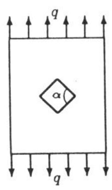


Figure 2. Dependence of the plate strength q on the crack length r



$$\sigma_{ij}(\rho, \theta) \sim K_1(y; \alpha)\rho^{-\gamma(\alpha)} f_{ij}(\theta; \alpha)$$

Figure 3. Plate with a diamond-shaped hole