NONLOCAL STRESS FIELD IN AN ORTHOTROPIC PLATE WITH A SLANTED CRACK

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We investigate within the framework of nonlocal elasticity the stress field of a unidirectional composite plate containing a crack orthogonal to the fibers direction, oriented at a given angle with the loading direction. The theory of nonlocal elasticity comprises the discrete nature of the material, and predicts a finite stress field at the crack tip. The anisotropic nature of the problem requires the use of a non-isotropic influence function that takes into account the fibers spacing. Results obtained are compared with results from classical elasticity, specifically the crack propagation direction as a function of the initial crack orientation. Of particular interest in the present work is the distance from the crack tip where the hoop stress is maximum, distance which cannot be predicted by classical elasticity. Extension of this work to cracks neither orthogonal nor parallel to the fibers direction is also discussed.

INTRODUCTION

The theory of classical elasticity, when applied to bodies containing a crack, gives stresses σ_x, σ_y and τ_{xy} that vary as the inverse square root of r where r is the distance to the crack tip. This implies that no matter how small the remotely applied stress is, the resulting stress field at the crack tip is infinite. The existence of this singularity remains a pitfall of classical elasticity. Unlike classical elasticity, the theory of nonlocal elasticity, developed by Eringen [1] following work from Kröner [2] and Kunin [3], takes into account the discrete nature of matter and incorporates the fact that forces among atoms are long-range ones. When applied to fracture mechanics, nonlocal elasticity theory predicts a finite stress field at the crack tip. Indeed, nonlocal effects become dominant at the crack tip since its characteristic dimension is on the order of an interatomic distance. This explains the breakdown of classical theories. Nonlocal elasticity has been applied to mode I [4,5], mode II [6], mixed-mode I-II [7], and mode III [8] loadings by Eringen in the case of isotropic materials. In this paper, we extend this analysis to anisotropic materials and investigate the nonlocal stress field in a unidirectional composite plate containing a crack orthogonal to the fibers direction, oriented at an angle varying from 0° to 90° with the loading direction. We then compare our results with results from classical elasticity [9].

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THE NONLOCAL ELASTICITY THEORY

We model the unidirectional fibers composite as an anisotropic linear elastic solid which will be considered homogeneous as a first approximation. Body forces are neglected and the problem is treated as a static one. The fundamental equations of nonlocal elasticity [1] are then given by equations (1-4), where ($\bar{r}$, $\bar{r}'$) are the

$$\frac{\partial t_{kl}(\bar{r})}{\partial x_k} = 0 \qquad (1)$$

$$t_{kl}(\bar{r}) = \int_{\mathcal{V}} \alpha(|\bar{r}' - \bar{r}|)\sigma_{kl}(\bar{r}')\cdot dv(\bar{r}') \qquad (2)$$

$$\sigma_{kl}(\bar{r}') = C_{klmn}\, e_{mn}(\bar{r}') \qquad (3)$$

$$e_{kl}(\bar{r}') = \frac{1}{2}\left[\frac{\partial u_k(\bar{r}')}{\partial x'_l} + \frac{\partial u_l(\bar{r}')}{\partial x'_k}\right] \qquad (4)$$

position vectors, σ is the local stress tensor, **t** is the nonlocal stress tensor, **e** is the strain tensor, $\bar{u}$ is the displacement vector, C_{klmn} is the anisotropic stiffness matrix, and (k,l,m,n) are indices for the three cartesian directions (x,y,z). Equations (2) are the nonlocal constitutive equations. The attenuation function $\alpha(|\bar{r}' - \bar{r}|)$ characterizes the behavior of the material, and as such should be representative of the material inner structure. For metallic materials for instance, atomic interactions decrease rapidly beyond the interatomic distance b, which leads to the concept of an influence region $\mathcal{R}$ of radius b around a point at $\bar{r}$. The attenuation function should be such that the dispersion curves obtained from the propagation of plane waves according to the displacement field equations (1-4) match the Born-Kármán lattice dynamics dispersion curves [6]. In order to be consistent, the nonlocal theory must also revert to the classical elasticity theory when b goes to 0, which is equivalent to having α revert to the delta Dirac distribution in the classical case. In addition the attenuation function must obviously be maximum at $\bar{r}' = \bar{r}$, and be normalized. Furthermore Bazant [10] pointed out that the Fourier transform of the attenuation function must be strictly positive. Acceptable functions include the normal distribution derived from one dimensional lattice dynamics and already been used by Eringen in the study of mode II [6] and mode III [7] fractures. In a subsequent study of mode I fracture [5], Eringen refined the theory by considering two dimensional lattice dynamics, and derived the following attenuation function:

$$\alpha(|\bar{r} - \bar{r}'|) = \frac{1}{2\pi\beta^2} K_0\left(\frac{1}{\beta}\sqrt{|\bar{r} - \bar{r}'|\cdot|\bar{r} - \bar{r}'|}\right) \qquad (5)$$

where K_0 is the modified Bessel function of the second kind of order zero, and β is a coefficient on the order of b that characterizes the attenuation. In the case of a unidirectional composite, we have two preferred directions, longitudinal (fibers direction) and transverse, hence our medium is orthotropic. Since we consider a crack along the ($\bar{x}$) axis orthogonal to the fibers, we modify the influence function (5) as in (6). The orthotropic properties of the composite are modeled by considering two different attenuation coefficients β_L and β_T, for the longitudinal (fiber) and transverse directions respectively. The attenuation function given by (6) has a strictly positive Fourier transform, is normalized and reverts to the delta Dirac

$$\alpha(|\bar{r}-\bar{r}'|)=\frac{1}{2\pi\beta_L\beta_T}K_0\left(\sqrt{\frac{(x'-x)^2}{\beta_T^2}+\frac{(y'-y)^2}{\beta_L^2}}\right) \tag{6}$$

distribution in the classical case [11]. Substituting equations (2), (3) and (4) into equation (1) and using the Green-Gauss theorem leads to the field equations (7).

$$\begin{aligned}&\int_{\mathcal{V}}\alpha(|\bar{r}'-\bar{r}|)\frac{C_{klmn}}{2}\left[u_{m,nk}(\bar{r}')+u_{n,mk}(\bar{r}')\right]\cdot dv(\bar{r}')\\&-\oint_{\partial\mathcal{V}}\alpha(|\bar{r}'-\bar{r}|)\left\{C_{klmn}e_{mn}(\bar{r}')\right\}\cdot da_k(\bar{r}')=0\end{aligned} \tag{7}$$

PLATE WITH SLANTED CRACK UNDER TENSION

The geometry for the problem studied herein is shown on Figure 1. The angle γ defines the crack initial orientation from the loading axis, and the angle θ defines the crack propagation direction from the crack initial axis. It will prove more convenient for the handling of boundary conditions to treat the problem in reference axes aligned with the crack. We thus perform an axis transformation from the loading axes $(\bar{x}_1,\bar{y}_1)$ to the crack axes $(\bar{x},\bar{y})$. Our initial boundary value problem now becomes the one of an infinite plate with a crack that is uniformly loaded in shear and tension by stresses σ_{yy}^0 and τ_{xy}^0 given by (8):

$$\sigma_{yy}^0=\frac{\sigma_a}{2}\left(1-cos(2\gamma)\right) \quad \text{and} \quad \tau_{xy}^0=\frac{\sigma_a}{2}sin(2\gamma) \tag{8}$$

The associated boundary conditions are:

$$\begin{cases} t_{yy}(x,0)=-\sigma_{yy}^0 \text{ and } t_{xy}(x,0)=-\tau_{xy}^0 & \text{for } |x|<a \\ u=u_x=u_1\rightarrow 0 \text{ and } v=u_y=u_2\rightarrow 0 & \text{when } \sqrt{x^2+y^2}\rightarrow\infty \\ \text{We consider the case of plane strain, hence } u_3=0. \end{cases}$$

Using the properties of the stress field and the linearity of the nonlocal equations, we can transform equations (7) into a system of partial differential equations for the displacements (u,v), and then prove that these equations together with the boundary conditions are equivalent to solving the classical elasticity problem [11]. The displacement field in classical and nonlocal elasticity are therefore identical, and we may use the classical stresses σ_{kl} in equations (2) for the calculation of the nonlocal stresses t_{kl}. The classical stress field σ_{kl} has been calculated and may be found in [9]. The nonlocal stresses t_{xx}, t_{yy} and t_{xy} are then, from relations (2), given by double integrals over the two space variables (x',y'). This renders their calculation by a numerical method very unreliable, especially for values of the attenuation coefficients β_L and β_T within the range of matrix (polymer or metallic) micro-structural length and fiber spacing respectively. It is nevertheless possible to obtain the nonlocal stresses under the form of a single integral over the phase variable ξ of a Fourier transform as a function of space coefficients (x,y), with β_L and β_T, crack length, material constants, and loading amplitude and angle as the only parameters [11]. They are given in condensed form by equations (9). The algebraic functions $\mathbf{K}^1_{xx}$, $\mathbf{K}^1_{yy}$, $\mathbf{K}^1_{xy}$, $\mathbf{K}^2_{xx}$, $\mathbf{K}^2_{yy}$, $\mathbf{K}^2_{xy}$ may be found in [11]. This allows the calculation of nonlocal stresses for various values of the parameters. Our solution is valid only in the vicinity of the crack tip since the relations giving the classical stresses [9] are valid only for $|x|$ and $|y|$ small before the crack length.

$$\begin{pmatrix} t_{xx} \\ t_{yy} \\ t_{xy} \end{pmatrix} = \frac{\sqrt{a}}{\beta_L\sqrt{2\pi}}\left\{\sigma^0_{yy}\int_0^\infty \frac{\begin{pmatrix}\mathbf{K}^1_{xx}\\ \mathbf{K}^1_{yy}\\ \mathbf{K}^1_{xy}\end{pmatrix}\begin{pmatrix}x,y,a,\\ \beta_L,\beta_T,\xi\end{pmatrix}}{\sqrt{\xi}\sqrt{1+\beta_T{}^2\xi^2}}\cdot d\xi + \tau^0_{xy}\int_0^\infty \frac{\begin{pmatrix}\mathbf{K}^2_{xx}\\ \mathbf{K}^2_{yy}\\ \mathbf{K}^2_{xy}\end{pmatrix}\begin{pmatrix}x,y,a,\\ \beta_L,\beta_T,\xi\end{pmatrix}}{\sqrt{\xi}\sqrt{1+\beta_T{}^2\xi^2}}\cdot d\xi\right\} \tag{9}$$

CRACK PROPAGATION DIRECTION

Numerical calculation of the indefinite integrals was done using the "Mathcad" software. Typical materials properties for a carbon/epoxy composite were taken from [9] and are given in Table 1, along with the loading parameters. The transverse attenuation coefficient β_T was chosen on the order of the distance between fibers, which is around ($0.001\cdot a$) for a 60% fiber volume fraction, a being the half crack length. The longitudinal attenuation coefficient β_L was chosen as ($a\cdot 10^{-6}$), corresponding to an interaction on the order of a few intermolecular or interatomic distances. To obtain the distance from the crack tip where the hoop stress is maximum, and the crack propagation direction, we first calculate the ratio R_{NL} of the nonlocal hoop stress $t_{\theta\theta}$ to the hoop fracture strength $S_{\theta\theta}$. Following [9], we define the hoop fracture strength by equation (10), where S_x and S_y are the fracture strength along the $\vec{x}$ (transverse) and $\vec{y}$ (fiber) directions respectively. Their values are taken from [9] and given in Table 1.

$$S_{\theta\theta} = S_x\cdot\cos(\theta)^2 + S_y\cdot\sin(\theta)^2 \tag{10}$$

TABLE 1. Loading and materials parameters

E_1 (Pa)	E_2 (Pa)	G_{12} (Pa)	ν_{12}	ν_{23}	a (cm)	γ	σ_a (Pa)	S_x (Pa)	S_y (Pa)
$70\cdot10^9$	$11\cdot10^9$	$5.65\cdot10^9$	0.24	0.3	1.25	45°	$7\cdot10^6$	$1.5\cdot10^9$	$43.8\cdot10^6$

Figure 3 gives the ratio R_{NL} as a function of the angle θ at the two characteristic distances β_L and β_T. On the same graph is plotted the ratio R_L of the local hoop stress $\sigma_{\theta\theta}$ to the hoop fracture strength $S_{\theta\theta}$ at the distance β_T. Nonlocal theory is in agreement with the classical theory and gives a propagation angle of θ_c=270° for a loading angle of γ=45°. This corresponds to a propagation along the fibers as expected [9]. The angle θ_c does not vary with the distance from the crack tip. Plots of R_{NL} for other loading angles γ between 0° and 90° also give a propagation angle θ_c of 270°. Examination of the ratio R_{NL} at θ_c as a function of r shown in Figure 2 clearly shows that the maximum hoop stress occurs not at the crack tip, but at a distance D equal to 7.8 times the influence length β_L. Figure 4 gives the distance D as a function of the loading angle γ in units of β_L. We notice that the distance D varies little, from (8.3)β_L at 0° to (6.4)β_L at 82.5°. This distance could not be predicted using classical theories since they do not comprise a length scaling.

CONCLUSION

The stress field in an orthotropic plate loaded along the fiber direction with a tensile normal and shear components has been investigated in the context of the nonlocal theory of elasticity. Results give predictions of the propagating angle which are in excellent agreement with the classical theory of elasticity and with experiments. A new feature brought forward by the nonlocal theory is the prediction of the distance at which the hoop stress is maximum in the direction of propagation. This distance is on the order of 8 times the characteristic microstructural length of the matrix. It would be interesting to extend this limited study to the general case of orthotropic plates with a crack at any angle from the fibers direction, and to compare it to both the classical results and experiments on composite materials. This general case however raises further mathematical difficulties for the calculation of the stress field. It will be investigated in a future work by the author. Even though nonlocal theory as used in this article models composite materials as homogeneous, and thus does not differentiate between matrix and fibers, it still constitutes an enticing alternative to the simpler classical theories. It is a first step in the right direction as it incorporates the fibers spacing. Nonlocal theories are however not limited to the treatment of homogeneous cases, and can be used to analyze a material with space-varying mechanical properties, even though such refinement is expected to lead to mathematical intricacies. In addition, when the crack is not aligned with one of the composite principal directions, crack surface displacement coupling occurs, and may cause crack closure [12]. This is an issue which has been overlooked by all previous papers on the modeling of crack propagation in composite plate (using classical elasticity), and which would need to be addressed.

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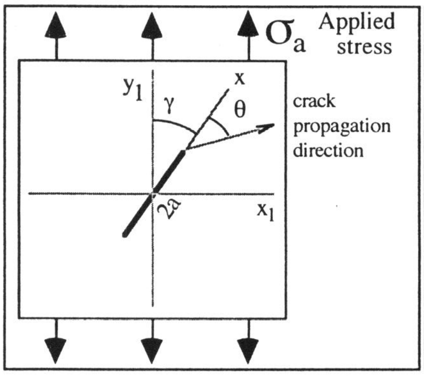

Figure 1. Cracked plate in tension.

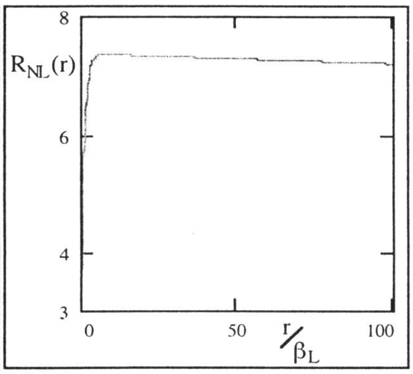

Figure 2. Nonlocal hoop stress at θ_c.

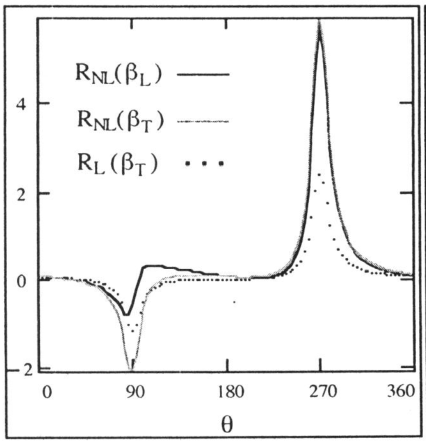

Figure 3. Nonlocal and local hoop stress

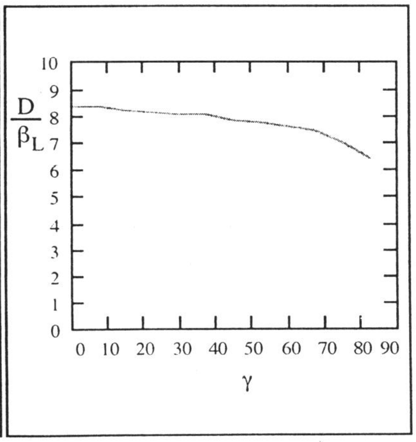

Figure 4. Normalized distance D vs. loading angle.