NON SINGULAR TERMS IN CRACK TIP STRESS DISTRIBUTION FOR PLANE PROBLEMS

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The effect of non singular terms in the elastic stress field near the crack tip for a two dimensional problem under Mode I loading is discussed. It is shown that the knowledge of the Weight Function and of the T-stress allows to evaluate the crack tip stress field with good accuracy not only on the ligament but in any position near the tip for any stress components. As an example, the central cracked panel under uniaxial tension is analysed and the effect of non singular terms in the shape of the plastic zone is shown.

INTRODUCTION

In the classical approach to Linear Elastic Fracture Mechanics (LEFM) the stress distribution around the crack tip is usually assumed to be defined by the Stress Intensity Factor (SIF or K), i.e. by the first -singular- term of the series expansion constituting the exact solution of the crack problem. However, experience recently gained on some special cases (quite important for practical applications as small or pressurized cracks) indicated that the SIF alone could be not sufficient to predict the crack behaviour. In those conditions, a more accurate characterization of the stress distribution nearby the crack tip is required which includes some non singular terms as shown by Sumpter and Forbes (1) and Romeo and Ballarini (2).

For a general fracture problem, such accuracy improvement can be obtained by numerical (e.g. Finite Element) models, which may be quite expensive and time consuming. In the present paper, a simple approximate method for evaluating the complete stress tensor for mode I loading is presented. The following five terms series expansion for the stress components near the tip is assumed:

$$\sigma_{ij} = \frac{K}{\sqrt{2\pi r}} f_{ij,0}(\theta) + a_1 f_{ij,1}(\theta) + a_2 \sqrt{r}\, f_{ij,2}(\theta) + a_3\, r\, f_{ij,3}(\theta) + a_4 \sqrt{r^3} f_{ij,4}(\theta) \qquad (1)$$

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where a_i and $f_{ij}(\theta)$ are scalar values and geometrical functions respectively to be determined, and other symbols are indicated in Fig. 1(a).

FUNDAMENTALS

Let us consider a plane body carrying a crack having length a under loads and boundary conditions producing Mode I loading only. The normal stress component acting on the ligament will be indicated as $\sigma_{yy}(x,a)$. The Weight Function (WF) Method allows to evaluate the SIF, $K(c)$, for any virtual crack of length c (in particular for $c \geq a$) integrating the WF $h(x,c)$ multiplied by the nominal stress acting in the uncracked body, subjected to the same loading and boundary conditions:

$$K(c) = \int_0^c \sigma_{yy}(x,0)\, h(x,c)\, dx \tag{2}$$

As observed by Beghini et al. (3), the SIF may also be obtained by integrating the stress distribution acting on the ligament ahead of the crack tip (Fig.1(b)):

$$K(c) = \int_a^c \sigma_{yy}(x,a)\, h(x,c)\, dx \tag{3}$$

Once the WF is known, the above eqn. (3) can be applied in order to evaluate the normal stress distribution on the ligament ahead of the crack. According to eqn. (1) (where r=x-a and θ=0), the following relationship holds:

$$\sigma_{yy}(x,a) = \frac{K(a)}{\sqrt{2\pi\,(x-a)}} + a_2\,\sqrt{(x-a)} + a_4\,\sqrt{(x-a)^3} \tag{4}$$

being $f_{yy,1}(\theta) = f_{yy,3}(\theta) = 0$ Substituting eqn .(4) in eqn.(3) the following equation can be obtained:

$$K(c) = \int_a^c \left[\frac{K(a)}{\sqrt{2\pi\,(x-a)}} + a_2\,\sqrt{(x-a)} + a_4\,\sqrt{(x-a)^3} \right] h(x,c)\, dx \tag{5}$$

where $K(c)$ is calculated by eqn.(2) and a_2 and a_4 are unknowns. Eqn. (5) can be solved in closed form or numerically depending on $h(x,c)$ by taking at least two crack lengths c (i.e.: c_1 and c_2). It may be observed that the crack length values, c_i, required for evaluating the unknown parameters must be adequately chosen with reference to the extension of the region ahead of the tip where the stress has to be reproduced. Hence, of the five terms indicated in eqn.(1), the first is determined by $K(a)$, the third and the fifth can be calculated by the method described above (i.e. by the knowledge of the WF). The remaining coefficients a_1 and a_3 are still unknown as they're independent from the σ_{yy} on the ligament. In particular, a_1 is related to a uniform stress field along the x axis (affecting the σ_{xx} component only) known as T-stress. For many geometries, several collections of T-stress values are available in literature, as those provided by Larsson and Carlsson (4) and Sherry et al. (5). Nothing can be said about the value of a_3 which produces no influence on σ_{yy} (being $f_{yy,3}(\theta) = 0$)

and affects the other stress components (being $f_{xx,3}(\theta) = \cos\theta$ and $f_{xy,3}(\theta) = \sin\theta$). On the contrary, owing to the expressions of $f_{ij,2}(\theta)$ and $f_{ij,4}(\theta)$, the knowledge of a_2 and a_4 allows the evaluation of the related stress components also in other positions near the crack tip, as illustrated in Fig.2.

A better description of σ_{yy} in the region near the crack tip by means of a limited number of terms is obtained by adding constant and linear terms to the expression (4):

$$\sigma_{yy} = \frac{K(a)}{\sqrt{2\pi\,(x-a)}} + b_1 + a_2\sqrt{(x-a)} + b_3\,(x-a) + a_4\sqrt{(x-a)^3} \qquad (6)$$

These two terms should be zero if a rigorous power expansion were considered, nevertheless they allow to better reproduce the local stress distribution and to improve the accuracy of stress field evaluation near the crack tip. The stress components associated to eqn.(6) may be expressed by the following equation:

$$\sigma_{ij} = \frac{K}{\sqrt{2\pi r}} f_{ij,0}(\theta) + b_1 g_{ij,1}(\theta) + a_1 f_{ij,1}(\theta) + a_2\sqrt{r}\, f_{ij,2}(\theta) + b_3 r\, g_{ij,3}(\theta) + a_4\sqrt{r^3}\, f_{ij,4}(\theta) \qquad (7)$$

where the term containing a_3 was neglected due to the impossibility to evaluate the coefficient. By the same method illustrated above it's possible to evaluate b_1, a_2, b_3 and a_4, i.e. by considering more than two crack lengths c in eqn. (5). In this case $f_{ij,m}(\theta)$ (with m=0,1,2,4) are the same functions which appear in eqn.(1) while b_1 and b_3 multiply two new functions $g_{ij,1}(\theta)$ and $g_{ij,3}(\theta)$ which were determined by solving the elastic problem for a semi-infinite body loaded on half boundary by constant and linear stresses respectively. By using the Muskhelishvili complex variable approach (6) it can be shown that the respective complex potentials are:

$$_1(z) = \frac{\pi - i\cdot\ln z}{2\,\pi} \qquad _3(z) = \frac{-i\cdot z}{2\,\pi}\left(\frac{1}{3} + i\,\pi + \ln z\right) \qquad (8)$$

where $z = r\,e^{i\theta}$ and i is the imaginary unit.

APPLICATIONS

With reference to a Central Cracked Panel (CCP) in tension, a comparison is made between the stress distribution due to the singular term and that obtained by eqn.(7). The example is related to a panel width of 80 mm carrying a crack having a total length $2a$=8 mm and the applied tension σ_0=1 MPa. The approximate WF proposed by Wu and Carlsson (7) is considered and the analysis is performed in the zone $a < c \leq 1.1\,a$. An FE analysis was used as reference. The accuracy of this numerical evaluation was tested by analysing the equivalent Griffith problem with a similar FE model. In this case the FE solution gives K and stress values with errors less than 0.1% as compared to the theoretical solution. In Fig. 3 a comparison between the stress evaluated by different methods near the crack tip for the CCP is shown. It can be observed that the proposed method differs from the reference solution of quantities comparable to the accuracy of FE solution. In Fig. 4 the loci of constant Mises stress σ_{eq} are plotted for two ratios σ_{eq} / σ_0 for plane stress and plane strain conditions. In those scales present solution is not distinguishable by the FE which is not reported for clarity. In this kind of diagram, it is possible to verify the solution accuracy by considering all the stress components and any position on the plane. These maps can be considered as an approximation of the near tip plastic zone for a material having the yield strength $\sigma_{ys} = \sigma_{eq}$. It can be observed that both shape and extension of the plastic zone is affected by the non singular terms particularly in plane strain conditions. It is worth noting that the discrepancy is more evident in plane strain in the range $\theta \in (\pi/4, 3\pi/4)$ than along the ligament where the singular term is predominant.

CONCLUSIONS

From a theoretical point of view, the power expansion of the function along the ligament for a Mode I loading crack has no constant and linear terms. This is true for any kind of loading conditions including uniaxial and bending loading. The stress distribution far from the crack results from all the other terms of the series expansion. If the stress has to be obtained in a region not far from the crack tip a five terms series expansion seems accurate enough to obtain any stress component. These terms can be obtained on the bases of the WF and the T-stress and for many problems this allows to avoid complex numerical analyses. The method was developed and applied to a Mode I loading, however its extension to Mode II is rather straightforward.

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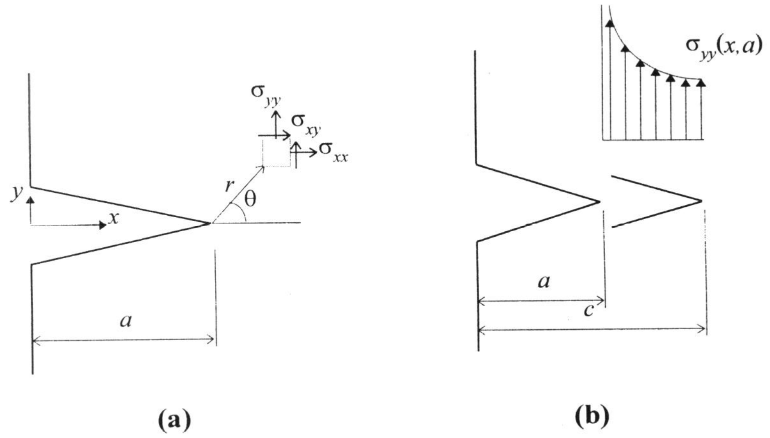

Figure 1 (a) Frame of reference and used symbols; (b) real a and virtual c crack and stress distribution on the ligament.

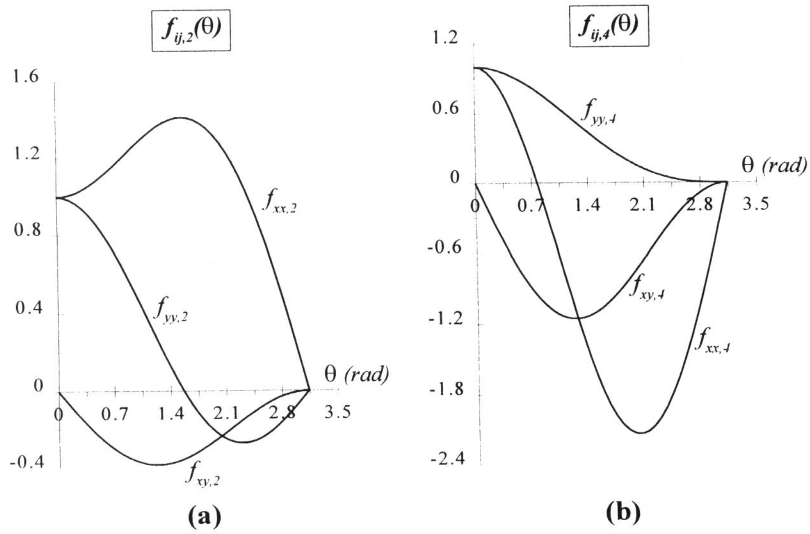

Figure 2 Graphical representation of the functions $f_{ij,2}(\theta)$ (a) and $f_{ij,4}(\theta)$ (b). Only $0 \leq \theta \leq \pi$ is considered because of the symmetry of the problem.

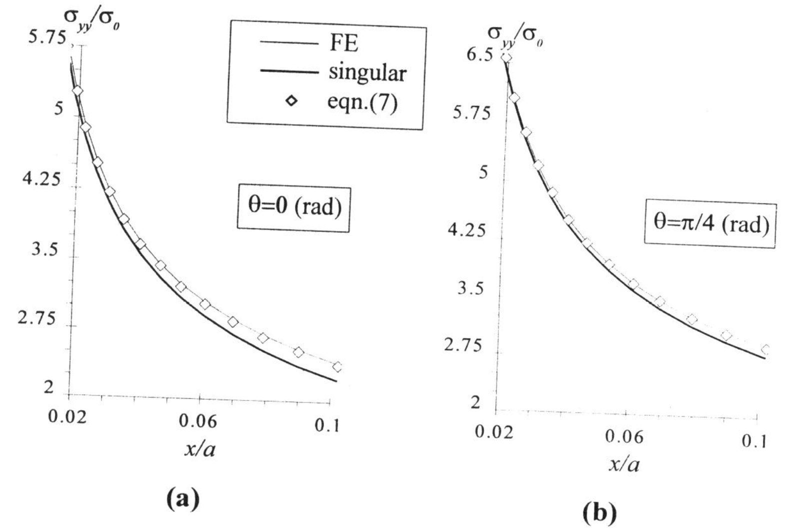

Figure 3 Comparison of the σ_{yy} evaluation using K, Finite Elements and eqn.(7) in corrispondence of two angles: (a) θ=0 and (b) θ=π/4 rad.

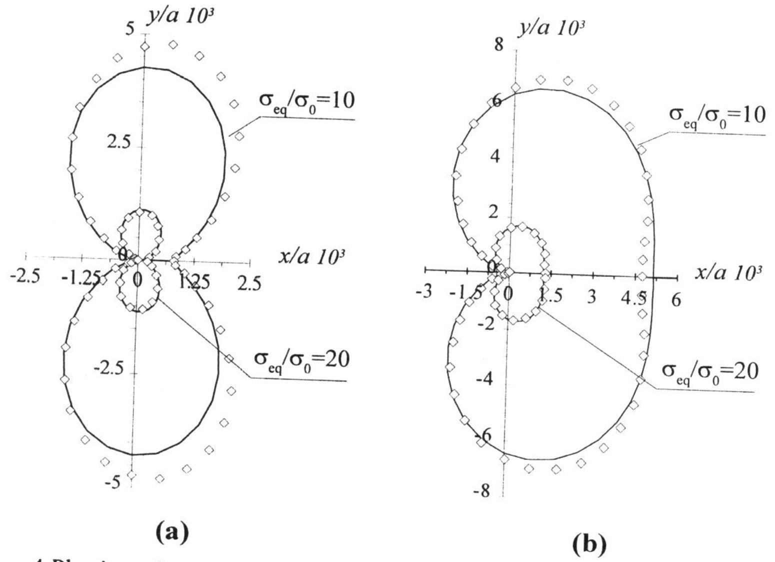

Figure 4 Plastic region in plane strain (a) and plane stress (b) conditions: diamonds indicate the present result and solid line the singular term only.