JOINING ELASTIC PLATES INTO A PACKAGE ALONG CURVES

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A package of thin infinite elastic plates laid over one another and joined together along curves is studied in terms of fracture mechanics. Specified forces applied to infinite points on the plates act in the plate planes. The plates are in a generalized plane stressed state. The forces are transferred from one plate to another only through the lines of junction of the plates, these differing in their elastic properties and thicknesses. The stressed state near the lines of junction of the plates is investigated. The stress intensity factors near the ends of these lines are found. Cases where plates are joined together exclusively along collinear segments or exclusively along a circumference are dealt with at length, an explicitly analytical solution of the problem being obtained for them.

INTRODUCTION

In many fields of engineering packages of plates are used that are joined together along narrow strips. For practical calculations those strips can be replaced by lines. One may proceed in a similar way for plates joined together with rivets spaced closely along narrow strips. Also, many multilayer composites with "soft" layers can be regarded as packages of plates joined together along a regular system of curves (e.g., a singly or doubly periodic system of straight lines).

This paper deals, in terms of integral equations and the Riemann boundary value problem, with the stressed state of a package of thin infinite elastic plates joined together along curves (joined, in particular, exclusively along collinear segments or exclusively along a circumference). The stresses are shown to have the same asymptotic representation in each plate near the ends of the open lines of junction as those near the vertex of a thin acute-

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angled rigid inclusion in a single plate. The law governing the stress intensity factors (SIFs) near those ends in terms of the initial data (specified forces, elastic constants, plate thicknesses) and the SIFs near the vertex of a thin acute-angled rigid inclusion in a single plate is established. Packages of plates joined together exclusively at individual points or exclusively along collinear segments were previously studied.

Junction along open curves

Assume that $E_1, E_2, \ldots, E_n$ are infinite thin homogeneous isotropic elastic plates coincident with the plane of a complex variable $z = x + iy$. Plate E_k has thickness h_k and is characterized by elastic constants $\mu_k, \kappa_k = (3 - \nu_k)/(1 + \nu_k)$, where μ_k is the shear modulus and ν_k is Poisson's ratio. The plates are superimposed, they all being joined together along curves $L_1, L_2, \ldots, L_m$. At point $z = \infty$ in plate E_k $(1 \le k \le n)$, specified stresses $(\sigma_x, \sigma_y, \tau_{xy})_k^\infty$, rotations ω_k^∞ and force $P_k = X_k + iY_k$ are in operation. All the specified forces act in the plate planes, their values being given on a plate unit thickness basis. There is a friction-free or no contact between the plate surfaces. The forces are transferred from one plate to another only through the lines of junction of the plates. The three-dimensional effects of the stress concentration along and near the lines of junction of the plates will be neglected.

In this case, a stressed state, which differs little from the generalized plane state, occurs in each plate E_k. That state is determined in terms of the Muskhelishvili (1) formulas via two functions (or complex potentials) $\Phi_k(z)$, $\Psi_k(z)$ which can be written, according to M.P. Savruk (2), as

$$\Phi_k(z) = \frac{1}{\pi i(1+\kappa_k)} \int_L \frac{q_k(t)}{t-z}\,dt + \gamma_k$$

$$\Psi_k(z) = \frac{1}{\pi i(1+\kappa_k)} \int_L \left(\frac{\kappa_k \overline{q_k(t)}}{t-z}\,\overline{dt} - \frac{\bar{t} q_k(t)}{(t-z)^2}\,dt \right) + \gamma_k' \tag{1}$$

$$\gamma_k = \frac{1}{4}(\sigma_x + \sigma_y)_k^\infty + \frac{2i\mu_k\omega_k^\infty}{1+\kappa_k},\ \gamma_k' = \frac{1}{2}(\sigma_y - \sigma_x)_k^\infty + i(\tau_{xy})_k^\infty,\ L = \bigcup_{k=1}^{n} L_k$$

where $2q_k(t)$ is equal to the jump of vector $(N + iT)_k$ of the normal and the tangential stress in plate E_k upon crossing line L. The value q_k is taken on a plate E_k unit thickness basis. The unknown function $q_k(t), k = \overline{1,n}$ must satisfy the conditions of equilibrium of the line L points:

$$\sum_{k=1}^{n} h_k q_k(t) = 0,\ t \in L; \quad \int_L q_k(t)\,dt = \frac{i}{2} P_k,\ k = \overline{1, n-1} \tag{2}$$

as well as those of mutual equality of the displacement of the points of all the plates along line L. These conditions will make it possible to obtain, in terms of Eq. (1) and the Muskhelishvili (1) formula, a set of singular integral equations which are written in matrix form as

$$\frac{1}{\pi i}\int\limits_{L}\left[A\left(\frac{2}{\tau - t} + M_1(\tau,t)\right)q(\tau)d\tau + BM_2(\tau,t)\overline{q(\tau)\,d\tau}\right] = f(t),\ t \in L \quad (3)$$

$$A = \|a_{kj}\|, B = \|b_{kj}\|, q(\tau) = \mathrm{col}\{q_1,\ldots,q_{n-1}\}, f(t) = \mathrm{col}\{f_1,\ldots,f_{n-1}\}$$

$$a_{kj} = \frac{h_j}{h_n} + \frac{\kappa_k\mu_n(1+\kappa_n)}{\kappa_n\mu_k(1+\kappa_k)}\delta_{kj},\quad b_{kj} = \frac{1}{\kappa_n}\left(\frac{h_j}{h_n} + \frac{\mu_n(1+\kappa_n)}{\mu_k(1+\kappa_k)}\delta_{kj}\right),$$

$$f_k(t) = \frac{1+\kappa_n}{\kappa_n}[\kappa_n\gamma_n - \bar{\gamma}_n - \bar{\gamma}_n'\frac{d\bar{t}}{dt} - \frac{\mu_n}{\mu_k}(\kappa_k\gamma_k - \bar{\gamma}_k - \bar{\gamma}_k'\frac{d\bar{t}}{dt})],\ k,j = \overline{1,n-1}$$

$$M_1(\tau,t) = \frac{d}{dt}\ln((\tau - t)/(\bar{\tau} - \bar{t})),\quad M_2(\tau,t) = -\frac{d}{dt}((\tau - t)/(\bar{\tau} - \bar{t}))$$

It follows from the theory of sets of singular integral equations developed by N.I. Muskhelishvili (3) that set (3) in the class of functions admitting of integrable singularities is always solvable and contains $m(n-1)$ arbitrary complex constants, these being determined from conditions (2) and those of displacement single-valuedness. Functions $q_k(t)$ near either end $t = c$ of line L are written as $q_k(t) = (t-c)^{-1/2}q_*(t)$, where $q_*(t)$ is a certain bounded function. What has just been said leads one to the following conclusions.

Near the ends of line L of junction of the plates into a package the complex potentials $\Phi_k(z)$, $\Psi_k(z)$ are asymptotically similar to those in the case of the second fundamental problem of the theory of elasticity for a plane with cuts. Asymptotic representations of the stresses near those ends in each plate will be the same as those near the vertex of a thin rigid acute-angled inclusion in an individual plate (see L.T. Berezhnitsky et al (4)). The SIFs $(k_1, k_2)_j$ near the end $t = c$ of line L in plate E_j can be found in terms of the formula

$$(k_1 - ik_2)_j = \pm\frac{2i\kappa_j}{1+\kappa_j}\sqrt{2\pi|t-c|}\,q_j(t),\ j = \overline{1,n} \quad (4)$$

where the plus sign is used for the initial points of line L, whereas the minus sign is for the final ones. The SIFs $(k_1, k_2)_j$ corresponding to any $n-1$ plate are linearly non-interdependent, while the SIFs corresponding to all n plates are related to each other linearly by $\sum\limits_{j=1}^{n} h_j(1+\kappa_j)(k_1 - ik_2)_j/\kappa_j = 0$.

If all the plates have the same Poisson's ratio, the stressed state in plate E_k $(1 \leq k \leq n-1)$ is a sum of the stressed state, which is constant throughout the plate

$$(\sigma)_k = (\sigma)_k^\infty - (\sigma)_k^*, \quad \sigma = (\sigma_x, \sigma_y, \tau_{xy}), \quad \omega = \omega_k^\infty - \omega_0^*$$

$$(\sigma)_k^* = \sum_{j=1}^{n-1} c_{kj}(\mu_n(\sigma)_j^\infty/\mu_j - (\sigma)_n^\infty), \; (\omega)_k^* = \sum_{j=1}^{n-1} c_{kj}\mu_n(\omega_j^\infty - \omega_n^\infty)$$

where c_{kj} are the elements of the kth row of matrix $C = A^{-1}$, and from the stressed state in an individual unit-thickness plate with thin rigid acute-angled inclusions along line L on which the displacements are zero, while at point $z = \infty$ stresses $(\sigma_x, \sigma_y, \tau_{xy})_k^*$, rotations ω_k^* and force P_k are in operation. The stresses in plate E_n are found from the formula

$$(\sigma)_n = (\sigma)_n^\infty - \sum_{k=1}^{n-1} h_k(\sigma - \sigma^\infty)_k/h_n$$

Junction along collinear segments. Here, the kernels $M_{1,2}(\tau, t) = 0$ in set (3), this falling into individual equations which are not related to each other:

$$\frac{2}{\pi i}\int_L \frac{q_k(\tau)}{\tau - t}\,d\tau = g_k(t), \; g_k(t) = \sum_{j=1}^{n-1} c_{kj} f_j(t), \; t \in L, \; k = \overline{1, n-1}$$

Solving this equation via (4) yield the SIFs.

Example 1. Assume that all the plates are absolutely identical and they are joined together exclusively along a segment $L = [-a, a]$. Only at point $z = \infty$ of plate E_1 does a stress σ act along (case 1) or orthogonal to (case 2) the line L of junction, while the remaining initial forces are zero. Then, $(k_1 - ik_2)_1 = \sigma(n-1)(3-\kappa)\sqrt{a}/(4n)$; $(k_1 - ik_2)_j = \sigma(\kappa - 3)\sqrt{a}/(4n)$; $j = \overline{2, n}$ in case 1 and $(k_1 - ik_2)_1 = \sigma(1-n)(\kappa+1)\sqrt{a}/(4n)$; $(k_1 - ik_2)_j = \sigma(\kappa + 1)\sqrt{a}/(4n)$; $j = \overline{2, n}$ in case 2.

Example 2. Assume that all the plates are absolutely identical and they are joined together exclusively along a segment $[-a, a]$, their number is even and at point $z = \infty$ in plates $E_1, E_2, \ldots, E_{2n-1}, E_{2n}$ forces $P, -P, \ldots, P, -P$, respectively, are in operation. Then, $(k_1 - ik_2)_j = (-1)^{j+1}\kappa P/(\pi(\kappa+1)\sqrt{a})$.

The problem of joining elastic plates to form a package along collinear segments was previously solved using the method of the Riemann boundary value problem by V.V. Silvestrov (5), V.V. Silvestrov and G.E. Chekmarev (6).

Junction along a circumference. Assume that all the plates are joined together exclusively along a curcumference L : $|z| = 1$. Here, the plate E_k stresses $(\sigma_r, \sigma_\theta, \tau_{r\theta})_k$ related to polar coordinates r, θ are found in terms of the Muskhelishvili (1) formulas via functions $\Phi_k(z)$, $\Omega_k(z)$. The functions are analytical both inside and outside the circumference L except points $z = 0, z = \infty$, in the vicinity of which they have specified representations. On the circvumference L the conditions of a certain matrix Riemann boundary value problem, which express the equilibrium of the points of the circumference L and the equality of the displacements of the points in all the plates on L. By solving this matrix problem using the method described in references (5) and (6) the complex potentials $\Phi_k(z)$, $\Omega_k(z)$ were obtained in the form of power functions, the stressed state of the plates was examined in close detail, and explicit formulas were obtained for stress values both inside and outside the circumference L. Specifically, the stresses were shown to be constant in each plate along any radius of circle $|z| < 1$, these being independent, throughout the circle, of forces P_k specified at infinitely removed plate points.

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