FRACTURE INSTABILITIES AND SCALE EFFECTS IN BRITTLE SOLIDS AND BRITTLE MATRIX FIBROUS COMPOSITES

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The structural behavior of quasi-brittle materials and brittle-matrix composites ranges from stable to unstable depending on material properties, structure geometry, loading condition and external constrains. In this paper the fracture behavior of a composite characterized by a bilinear cohesive law is analyzed by means of a cohesive-crack model and a bridged-crack model. It is shown that the complex changes in the shape of the load-deflection curve for a three-point bending beam are controlled by two dimensionless parameters. The first, $s_E = \mathcal{G}_F/\sigma_u h$, depends on the beam depth, h, and on the composite fracture toughness, $\mathcal{G}_F$, and tensile strength, σ_u. The second, $\mathcal{G}_b/\mathcal{G}^m_{IC}$, is the ratio of the energy necessary to develop the bridging mechanism of the secondary phases, $\mathcal{G}_b$, to the intrinsic fracture energy of the matrix, $\mathcal{G}^m_{IC}$. It fundamentally depends on the shape of the cohesive law.

INTRODUCTION

The structural response of quasi-brittle materials and brittle matrix composites is not physically similar when the size scale of the body is varied. The nature of the crack and the structure behavior can range from stable to unstable depending on material properties, structure geometry, loading condition and external constrains. The two limiting solutions, given by Linear Elastic Fracture Mechanics and by the perfectly plastic limit analysis may be used only for extremely brittle cases (low fracture toughness, high tensile strength, large sized structures, initially uncracked specimens) and for extremely ductile cases, respectively.

This complex behavior arises due to the existence of inhomogeneities (voids, flaws, coarse grains, particles, aggregates, fibers,...) in the brittle matrix. These inhomogeneities control the crack propagation process, acting both in the wake of the macrocracks, where they bridge the faces of the crack, and ahead of the macrocracks, where they give rise to a microcracked process zone.

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In accordance with the model proposed by Barenblatt (1), the process zone is represented by a fictitious crack, and the bridging mechanisms by a distribution of closing tractions, σ. A relation $\sigma(w)$ links the closing tractions to the crack face opening, w. Three parameters characterize this relation and the bridging mechanisms of different composites: the shape of the law, the maximum traction, σ_u, and the critical crack opening, w_c, beyond which the closing tractions vanish.

The toughness peculiar to the matrix can be modeled as an intrinsic fracture energy[1], $\mathcal{G}^m_{IC}$, and the crack tip stress field assumed to be singular. These assumptions define the *bridged-crack model*, and imply that the closing tractions are related only to the secondary phases of the composite. The area beneath the bridging law is the energy necessary to develop the bridging mechanism, $\mathcal{G}_b$.

On the other hand, the crack tip stress field can be assumed to be finite, and the closing tractions to represent the homogenized composite, in accordance to the *cohesive-crack model.* The area beneath the cohesive law represents the composite fracture energy, $\mathcal{G}_F$, i.e., the energy necessary to produce a unit crack surface. Note that $\mathcal{G}_F = \mathcal{G}_b + \mathcal{G}^m_{IC}$ (see Cox and Marshall (2) for a review of the models).

Bridging mechanisms operating at very different scales are modeled based on these assumptions. In the modeling of the atomic bonds, the orders of magnitude of the two parameters of the cohesive law are $\sigma_u \approx 10^4$ *MPa* and $w_c \approx 10^{-4}$ μm; the bridging law of carbon fibers pulling-out from a ceramic matrix is characterized by $\sigma_u \approx 10^3$ *MPa* and $w_c \approx 10^1$ μm, and that of ductile particles by $\sigma_u \approx 10^1$ *MPa* and $w_c \approx 10^1$ μm; the cohesive law of a concrete is defined by $\sigma_u \approx 10^0$ *MPa* and $w_c \approx 10^{-1}$ *mm*, and if the concrete matrix is reinforced with steel fibers, by $w_c \approx 10^1$ *mm*.

The size of the bridged zone of a crack propagating in small-scale bridging (the LEFM condition), in an uniformly loaded infinite medium, gives a first indication of the behavior expected in fracture. For a rectilinear cohesive law and a vanishing intrinsic fracture toughness, $\mathcal{G}^m_{IC}=0$, this length scale is given by $a_s=(\pi/8)w_c E/\sigma_u^2$, E being the composite Young's modulus, Bilby et al. (3). For different cohesive laws the factor $\pi/8$ changes, but it remains of the same order of magnitude, Smith (4). Note that a_s coincides with the plastic zone size of Dugdale's model (5), and that is proportional to Hillerborg's characteristic length, $l_{ch}=w_c E/\sigma_u^2$, (6). If the matrix fracture toughness is nonvanishing, $\mathcal{G}^m_{IC} \neq 0$, the bridged zone size, a_s, progressively decreases for decreasing values of the ratio $\mathcal{G}_b/\mathcal{G}^m_{IC}$, Cox and Marshall (2).

Comparing the order of magnitude of a_s and the width h of a finite specimen, indicates whether the crack will approach the small-scale bridging limit configuration, $a_s \ll h$, or it will propagate under large-scale bridging conditions, $a_s \approx h$. In the first case the problem can be studied by means of LEFM, while in the

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[1] or equivalently as a critical crack tip stress intensity factor, $K^m_{IC} = (\mathcal{G}^m_{IC} E)^{0.5}$, E being the composite Young's modulus.

second case detailed calculations are required, and the behavior will be strongly affected by the geometry and the external loading. If this is the situation, a complete and unitary description of the behavior of brittle matrix composites, must rely on parameters characterizing the structure and not only the material.

Based on the assumptions of the cohesive-crack model, Carpinteri (7) defined a dimensionless parameter, which synthetically controls the fracture behavior of elasto-softening materials and the size-scale transition from ductile to brittle response of self similar structures. This parameter, called the brittleness number, $s_E = \mathcal{G}_F/\sigma_u h$, depends on both the material properties and the structure dimensions (see also Bosco and Carpinteri (8)).

For a linear softening cohesive law, it has been shown that the flexural behavior of a beam varies from ductile to brittle when s_E decreases. A truly brittle failure, and the classical LEFM instability, are found if $s_E = \mathcal{G}_F/\sigma_u h \rightarrow 0$, i.e., for low fracture toughness, high tensile strength and/or large structure size-scales. A high slenderness and the absence of initial notches favour this behavior.

If the composite material is characterized by a nonvanishing intrinsic fracture toughness, $\mathcal{G}^m_{IC} \neq 0$, the ratio $\mathcal{G}_b/\mathcal{G}^m_{IC}$, along with s_E and the shape of the bridging law, govern the structural behavior. Different size-scale transitions are found depending on $\mathcal{G}_b/\mathcal{G}^m_{IC}$, Carpinteri and Massabò (9).

In this paper a nonlinear fracture mechanics model, formulated by the authors for the analysis of brittle-matrix composites with uniformly distributed reinforcement, is briefly recalled, Carpinteri and Massabò (10). The two assumptions, of a vanishing and a nonvanishing crack tip stress intensity factor (cohesive-crack and bridged-crack), are examined in the simulation of the fracture behavior of a Three-Point Bending beam. A bilinear cohesive law is assumed to describe brittle-matrix composites with secondary phases pulling-out from the matrix. The influence of the ratio $\mathcal{G}_b/\mathcal{G}^m_{IC}$ on the size-scale effects is studied, and the limitations on the applicability of the bridged-crack model to structural analyses are highlighted.

THE THEORETICAL MODEL

Consider the schematic description of the cracked cross section of a composite beam, subjected to a bending moment M, and shown in Fig.1.a. The beam depth and thickness are h and b, and the crack length is a. In accordance with Barenblatt's model (1), a fictitious crack, of length a_f, acted upon by a continuous distribution of closing tractions, σ, represents the composite process zone. An assigned relationship, $\sigma(w)$, links the closing tractions to the crack opening displacements, w. The maximum traction and the critical crack opening are σ_u and w_c, respectively. The matrix is assumed to be linear-elastic, and reference is made to the two-dimensional single-edge notched-strip solutions, Tada et al. (11), to define the fracture mechanics parameters.

The constitutive flexural relationship, which characterizes the mechanical response of the component, is evaluated by following the evolution of the crack up to the total disconnection of the beam.

At the tip of the crack a global stress intensity factor, $K_I=K_{IM}+K_{I\sigma}$, is defined by means of the superposition principle. K_{IM} and $K_{I\sigma}$ are the stress intensity factors due to the applied bending moment, M, and to the closing tractions, σ, respectively.

The two assumptions of a singular and a finite crack tip stress field, previously outlined in the introduction, are examined, and two different crack growth criteria are consequently applied. In accordance with the *bridged-crack model*, the crack tip stress field is singular, and the crack growth criterion states $K_I= K^{m}_{IC}$, where K^{m}_{IC} is the matrix intrinsic fracture toughness. In this case the closing tractions represent the bridging mechanisms developed by the secondary phases (*bridging tractions*). On the other hand, in accordance with the *cohesive-crack model*, the stress field in the crack tip vicinity is finite, and the crack growth criterion states $K_I=0$. The closing tractions represent the combined restraining action of the matrix and the secondary phases on crack propagation (*cohesive tractions*).

By means of the crack propagation criteria, the crack-propagation moment, M_F, is derived for each crack length, and the corresponding localized rotation, φ, is calculated using Clapeyron's Theorem. If σ_u and h are chosen as the basic set of dimensionally independent variables, the nondimensional forms of the crack-propagation moment and the corresponding rotation are:

$$\frac{M_F}{\sigma_u h^2 b} = \frac{1}{Y_M(\xi)}\left\{\int_{\xi_r}^{\xi}\frac{\sigma(w(\zeta))}{\sigma_u}Y_P(\xi,\zeta)d\zeta+\sqrt{K}\right\} \quad (1)$$

$$\frac{\varphi}{\varepsilon_u} = 2\frac{M_F}{\sigma_u h^2 b}\int_0^{\xi}Y_M^2(\zeta)d\zeta - 2\int_{\xi_r}^{\xi}\int_{\zeta}^{\xi}Y_M(y)\,Y_P(y,\zeta)dy\frac{\sigma(w(\zeta))}{\sigma_u}d\zeta \quad (2)$$

where $\xi=a/h$ and $\xi_r=a_r/h$ are the normalized lengths of the crack and the traction-free crack (Fig.1), and Y_M and Y_P are polynomial functions related to the geometry of the specimen. The parameter K takes on the form:

$$K = \frac{s_E}{\varepsilon_u}\left(\frac{1}{1+\mathcal{G}_b / \mathcal{G}^{m}_{IC}}\right) \quad (3)$$

where $\mathcal{G}_b$ is the area beneath the bridging curve, $\mathcal{G}^{m}_{IC}$ is the matrix intrinsic fracture energy, $s_E=\mathcal{G}_F/\sigma_u h$ is the brittleness number, $\mathcal{G}_F$ being the composite fracture energy, and ε_u the ultimate elastic strain of the composite. Note that $K=0$ if $\mathcal{G}^{m}_{IC}=0$ (cohesive-crack model).

Eqs. (1) and (2) define a nonlinear statically indeterminate problem, the indeterminate closing tractions, $\sigma(w)$, depending on the unknown crack opening displacements, w. A numerical model has been formulated to evaluate the beam configuration satisfying both equilibrium condition and kinematics compatibility.

The analytical formulation brings out that, once fixed the shape of the bridging or cohesive law, the constitutive flexural behavior of self similar beams is governed by a different number of nondimensional parameters, depending on which of the two models is applied. For the assumed basic set of variables, the two dimensionless parameters of the *bridged-crack model* are s_E/ε_u and $\mathcal{G}_b/\mathcal{G}_{IC}^m$, while the sole dimensionless parameter of the *cohesive-crack model* is s_E/ε_u.

SCALING TRANSITIONS IN BRITTLE-MATRIX COMPOSITES

The analysis of a brittle matrix composite by means of the *cohesive-crack model* (vanishing stress intensity factor) requires the definition of the cohesive law, $\sigma(w)$, representing the homogenized material. This law is inferred from experimental measurements or from micromechanical modeling.

The bilinear law shown in Fig.1.b is a simple and representative approximation to the cohesive law of various composites. Depending on the location of the knee point, $k(\beta w_c, \alpha\sigma_u)$, brittle matrices reinforced with short fibers, hard or ductile particles or aggregates, are represented. The area beneath the curve defines the composite fracture energy, $\mathcal{G}_F$. The first and the second branch of the curve describe the bridging mechanisms peculiar to the matrix and developed by the fibers, respectively.

An alternative approach to study the fracture behavior of brittle-matrix composites, characterized by a bilinear cohesive law, is the application of the *bridged-crack model* (nonvanishing stress intensity factor). In this case, the energy dissipated in the first part of the cohesive law, which is usually small in relation to the total, is lumped into an intrinsic fracture energy, $\mathcal{G}_{IC}^m$. The matrix is assumed to be perfectly brittle, and the bridging mechanism of the secondary phases is described by the linear bridging law shown in Fig.1.c. The area beneath the curve, $\mathcal{G}_b$, represents the contribution of the reinforcement to the composite fracture energy, $\mathcal{G}_F=\mathcal{G}_b+\mathcal{G}_{IC}^m$. Based on these assumptions, the ratio $\mathcal{G}_b/\mathcal{G}_{IC}^m$ is kept unchanged so that one of the two dimensionless parameters of the model is fixed.

The two theoretical models may converge or diverge in the results, depending on the properties of the material and the size of the structure.

Consider a composite characterized by a cohesive law with $\alpha=0.2$ and $\beta=0.001$. This law approximates the behavior of a cementitious composite reinforced with short steel fibers, Carpinteri and Massabò (10). Note that the energy dissipated in the first portion of the law is very small with respect to the total, and $\mathcal{G}_b/\mathcal{G}_{IC}^m \approx 250$.

A Three-Point Bending beam, of depth h, thickness b, length $l=6h$, is analyzed by means of the two models. An ultimate elastic tensile strain $\varepsilon_u=1.5\times10^{-4}$ is assumed. The evolution of crack propagation from an initial notch, $a_0=0.1h$, up to

a=0.9h, is studied. The flexural response of the beam, in terms of dimensionless load, $Pl/\sigma_u h^2 b$, versus normalized mid-span deflection, δ/h, is controlled by the brittleness number $s_E=\mathcal{G}_F/\sigma_u h$. The dimensionless curves of Fig.2, describe variations of s_E over four orders of magnitude.

When $s_E \rightarrow 0$ the models predict the classical LEFM snap-back instability. The smallest s_E here examined, $s_E=3\times10^{-6}$, defines a transitional configuration, with a strain softening branch whose slope is nearly infinite. In the range of s_E varying from $s_E=3\times10^{-6}$ to $s_E=3\times10^{-4}$, the behavior changes from strain-softening to strain-hardening. The response is almost perfectly plastic for $s_E=2\times10^{-3}$, and then, for s_E varying from $s_E=2\times10^{-3}$ to $s_E=6\times10^{-4}$, it turns again strain-softening, and a loading capacity higher than the ultimate loading capacity at total disconnection is predicted (hyper-strength phenomenon).

Over the examined range of s_E, the bridged-crack model and the cohesive-crack model predict approximately the same results. For higher values of s_E the results diverge (see $s_E=3\times10^{-2}$). The cohesive-crack model predicts a progressive transition towards ductile responses and the plastic collapse. The behavior is similar to that predicted for a linear softening law, Carpinteri (7). On the other hand, the bridged-crack model predicts strain-softening responses, and the loading capacity of the beam is defined by the LEFM solution for a notch of depth a_0 and a fracture energy $\mathcal{G}_{IC}^{m}$.

This inconsistency is explained by the fact that the behavior over the last range of s_E is strongly matrix dominated, and the influence of the fibers decreases by increasing s_E. The bridged-crack model assumption of a perfectly brittle matrix, is not valid anymore, and a proper modeling of the bridging mechanisms peculiar to the matrix is necessary. This result confirms the well known conclusion that LEFM is not applicable for the description of quasi-brittle materials, for high fracture toughness, low tensile strength or small sized structures.

Note that the value of s_E for which the model results start diverging depends on both the initial notch length and the location of the knee point in the cohesive law. In the limit case of an unotched material, the bridged crack model predicts an unrealistic infinite loading capacity.

The nature of crack propagation changes according to the variations in s_E. For $s_E \rightarrow 0$ the crack propagates in small-scale bridging, and the response is controlled by the composite fracture energy $\mathcal{G}_F$, through the LEFM solution. For the intermediate values of s_E (diagrams a-d) the crack propagates in large scale bridging, and the behavior is controlled by the shape of the last branch of the cohesive law (bridging mechanisms of the reinforcement). For higher values of s_E, the crack propagates in large-scale bridging and it is fully crossed by the fibers up to total disconnection of the beam. The behavior is essentially controlled by the shape of the first branch of the cohesive law (bridging mechanism of the matrix).

The three curves for $s_E=6\times10^{-3}$, $s_E=2\times10^{-3}$, and $s_E=3\times10^{-4}$, illustrate the flexural responses usually observed from experimental tests on fiber reinforced cementitious materials. Consider for instance a fiber-reinforced concrete, characterized by $\sigma_u=6Mpa$, $w_c=8mm$, $\mathcal{G}_{IC}^{m}=0.02Nmm^{-1}$, and $\mathcal{G}_F=4.8Nmm^{-1}$. The three previously mentioned curves would describe the behavior of three beams of depths $h\cong130mm$, $h\cong400mm$, and $h\cong2600mm$, respectively. Note that the two theoretical models predict similar behavior over this interval of s_E.

Consider now a cohesive law with $\alpha=0.15$ and $\beta=0.1$. This law is used to represent a quasi-brittle material, e.g., plain concrete. Due to the different location of the knee point, the energy dissipated in the first portion of the law is very similar to the total, and $\mathcal{G}_b/\mathcal{G}_{IC}^{m}\approx2$.

The diagrams of Fig.3 show the dimensionless load-vs.-deflection curves obtained for a three-point bending beam, with the same geometry of the previously examined beam, and $\varepsilon_u=1\times10^{-4}$, when the parameter s_E is varied. The curve corresponding to the lowest s_E is characterized by a snap-back instability, with a softening branch of positive slope.

The cohesive-crack model predicts flexural responses typical of a quasi brittle material. The behavior changes from stable to unstable by decreasing s_E. The bridged crack model predicts similar results for s_E lower than $s_E=3\times10^{-5}$, but again, for higher values of s_E, the results diverge and the inapplicability of the model is manifest.

Consider a concrete characterized by $\mathcal{G}_F=0.1Nmm^{-1}$, $\sigma_u=5Mpa$ and $w_c=0.16mm$. The three curves for $s_E=3\times10^{-4}$, $s_E=9\times10^{-5}$, and $s_E=3\times10^{-5}$, would represent the behavior of three beams of depths $h\cong70mm$, $h\cong220mm$, and $h\cong650mm$, respectively. The assumption of lumping the energy dissipated in the first part of the cohesive law into an intrinsic fracture energy $\mathcal{G}_{IC}^{m}$, is therefore unacceptable in the simulation of experimental tests on quasi-brittle materials.

It is worth noticing that the diagrams of Fig.3 do not show the intermediate brittle to ductile transition, previously observed in Fig.2 for $s_E=3\times10^{-6}\div2\times10^{-3}$, which is typical of fibrous composites. This behavior is a consequence of the lower value assumed by the ratio $\mathcal{G}_b/\mathcal{G}_{IC}^{m}$ (see Carpinteri and Massabò (9), for a description of this behavior in a composite with a rectilinear bridging law).

CONCLUSIONS

The flexural behavior of fibrous brittle-matrix composites depends on the size-scale of the structure, besides the fracture toughness and the tensile strength of the material, through the parameter $s_E=\mathcal{G}_F/\sigma_u h$. A transition from brittle to ductile responses is predicted when the brittleness number s_E is increased.

A second parameter, which governs the structural behavior, is the ratio $\mathcal{G}_b/\mathcal{G}_{IC}^{m}$, between the contribution of the reinforcement, $\mathcal{G}_b$, and matrix, $\mathcal{G}_{IC}^{m}$, to the global fracture energy of the composite, $\mathcal{G}_F$. If the brittleness number s_E is kept unchanged, more ductile responses are predicted by increasing $\mathcal{G}_b/\mathcal{G}_{IC}^{m}$. A low matrix fracture toughness and a strong bridging mechanism of the reinforcement lead to this result.

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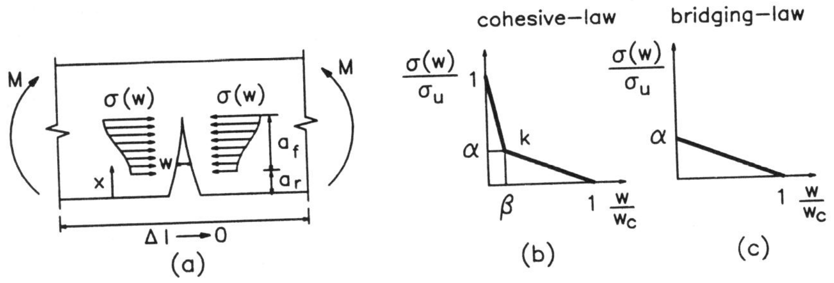

Figure 1: a) Schematic of the composite cross section; b) bilinear cohesive law; c) linear bridging law.

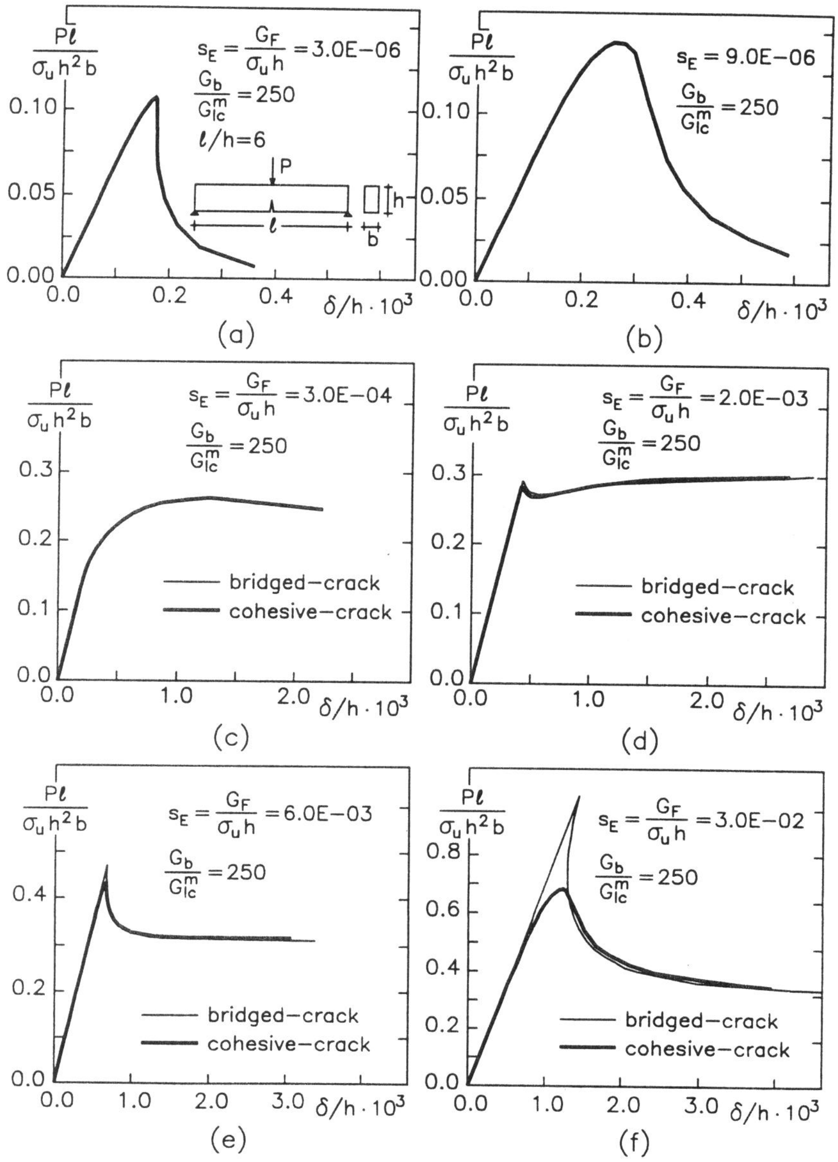

Figure 2: Dimensionless load-vs.-deflection curves for a three-point bending beam. Comparison between the bridged-crack model and the cohesive-crack model results. Bilinear cohesive law, α=0.2 and β=0.001, $\varepsilon_u=1.5\times10^{-4}$, $l=6h$.

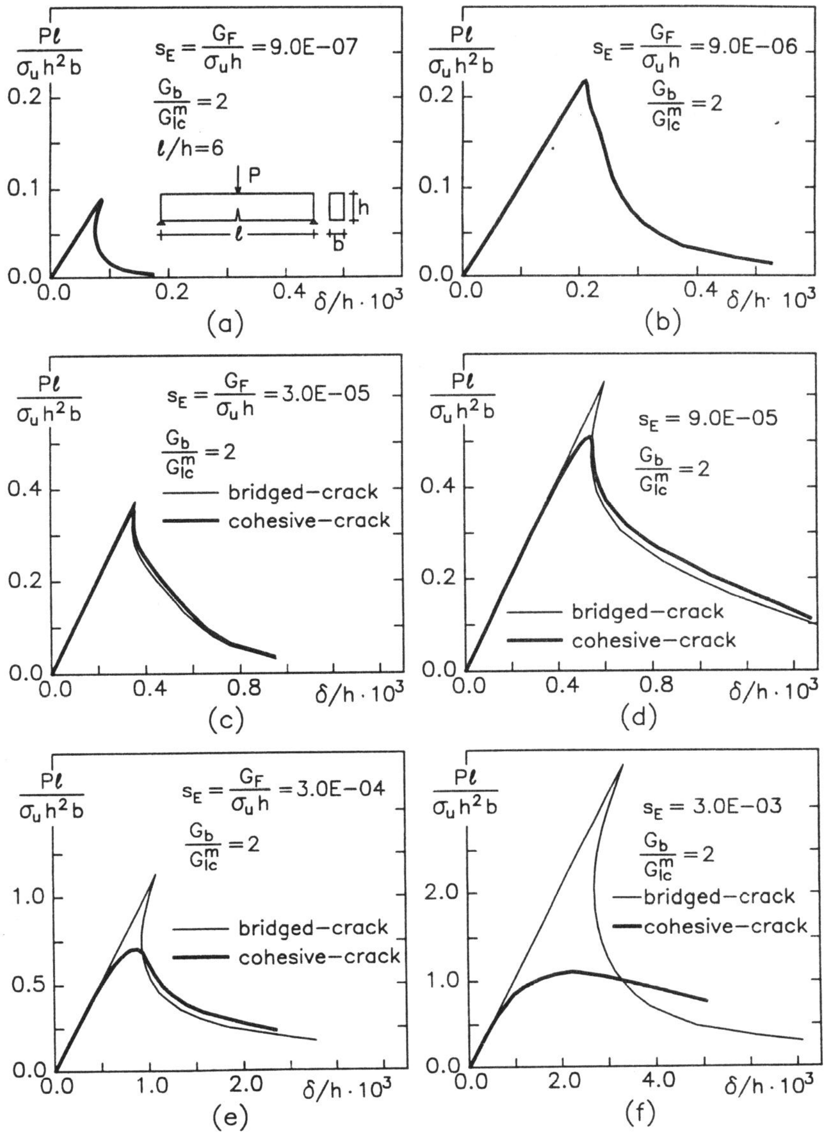

Figure 3: Dimensionless load-vs.-deflection curves for a three-point bending beam. Comparison between the bridged-crack model and the cohesive-crack model results. Bilinear cohesive law, α=0.15 and β=0.1, $\varepsilon_u=1\times10^{-4}$, $l=6h$.