CRACK TIP PLASTICITY INVESTIGATIONS USING DUGDALE STRIP YIELD MODEL APPROACH

D. Pustaić* and B. Štok†

The mechanical response of an infinite thin plate with a central line crack, which undergoes due to the applied remote biaxial loading plastic deformation in the neighbourhood of the crack tips, is considered in this paper. To describe the plastification process the Dugdale-Barenblatt yield model, which was originally used by Dugdale to solve a specific loading case in conjunction with the Tresca yield criterion, is utilized. Two yield criteria are considered, actually the ones attributed to Tresca and Mises. The analysis of the investigated mechanical response, which includes stresses and displacements, is based on its analytical determination using methods of complex analysis. At the end, the impact of the load biaxiality on the plastic yielding is discussed in view of the two considered yield criteria.

DUGDALE'S APPROACH FOR THE CRACK TIP PLASTICITY

The stress solution of an elastic crack problem is characterized by the crack tip singularity that cannot be withstood by the material elastic resistance. The material response is therefore inevitably accompanied by plastic deformation the extent of which may be, as it has been verified by experimental observation, closely estimated by the solution methods for elastic crack problems.

The crack tip plasticity analysis can be performed according to the premises of the Dugdale-Barenblatt yield model, Rice (1). In this investigation the Dugdale yield model, Dugdale (2), which is in fact a simplification of the more complex Barenblatt model will be assumed. When posing his hypothetical model, Dugdale assumed first that plastic deformation is governed by perfect plasticity under the supposition of constant cohesive stress $\sigma_{yy} = \sigma_o$ in the yielded area, with σ_o being the yield strength, and second that yielding of a material is confined to a narrow

* Faculty of Mechanical Engineering and Naval Architecture, University of Zagreb, Zagreb - Croatia
† Faculty of Mechanical Engineering, University of Ljubljana, Ljubljana - Slovenia

strip band, extending ahead from the crack tip and lying along the crack direction. Accordingly, he postulated the existence of an imaginary elastic crack composed of a physical blunt crack of length $2a$ and a supplementary cracked zone extended ahead at both tips of the virgin sharp crack for a distance r_p, the length of the supplementary crack being equal to the length of the plastic zone around the crack tip. The determination of stresses in the yielded plate is obtained as a superposition of two elastic responses, both taking the imaginary crack of length $2b$ into account. Actually, the elastic response due to the external loading of the modified cracked plate is superposed by the elastic response due to the application of the cohesive stresses. Because of the assumed elastic approach both responses are characterized by the stress singularity, their intensities being given by the stress intensity factors K_{ext} and K_{coh}, respectively. But, since in reality the stress singularity, introduced by the elastic approach, does not occur due to plastic yielding it has to be cancelled by imposing

$$K = K_{ext} + K_{coh} = 0 \quad . \tag{1}$$

The fulfilment of the above condition yields the plastic zone length r_p.

PROBLEM DEFINITION AND GOVERNING EQUATIONS

An infinite plate ($z \in D$, D: $|z| \geq 0$) with an embedded straight crack of length $2a$ lying on the x-axis ($z \in L$, L: $Re\ |z| \leq a$, $Im\ z = 0$), its material being supposed to exhibit elastic-perfectly plastic behaviour, is considered. An in-plane remote loading is assumed to be applied symmetrically in respect of the x- and y-axes, while the crack boundary is traction free. By adopting Dugdale's approach for the crack tip plasticity the original elastic-plastic boundary value problem that is defined on the domain D cut along the line L has to be adequately modified. The modified problem is an elastic one and is defined on the domain D^* ($z \in D^*$, D^*: $|z| \geq 0$) cut along the line L^* ($z \in L^*$, L^*: $Re\ |z| \leq b = a + r_p$, $Im\ z = 0$).

Since in this investigation we are particularly interested in the remote loading which is biaxially dependent, the boundary conditions at infinity read

$$\sigma_{xx}(z) = \sigma_{xx}^{\infty} = k\sigma_{\infty} \ , \quad \sigma_{yy}(z) = \sigma_{yy}^{\infty} = \sigma_{\infty} \ , \quad \sigma_{xy}(z) = 0 \qquad as\ |z| \to \infty \tag{2}$$

where σ_∞ ($\sigma_\infty > 0$) and k are real constants. On the edges of the imaginary crack, i.e. on the line L^*, the correspondent boundary conditions are as follows

$$\begin{aligned} &\sigma_{yy}(z) = \sigma_{xy}(z) = 0 && for\ z \in L \ , \\ &\sigma_{yy}(z) = \sigma_Y \ , \quad \sigma_{xy}(z) = 0 && for\ z \in L^* - L \ . \end{aligned} \tag{3}$$

Here, symbol σ_Y is introduced to denote the constant cohesive stress along the yielded zone, its magnitude not being necessary equal to the yield strength σ_o.

Governing equations of the modified problem, which can be considered as a plane stress problem, are those of the linear theory of elasticity in conjunction with any yield criterion, the latter governing the evolution of plastic yielding and implicitly affecting the magnitude of the cohesive stress σ_Y. In order to obtain analytical solution we adopt the methodology of complex analysis, Muskhelishvili (3), and look for analytic functions of the complex variable $z = x + i.y$ that fulfill both, boundary conditions (2) and (3) as well as equations governing the problem.

Considering the fact that due to symmetry the shear stress vanishes when $Im\, z = 0$, the governing equations of the plane theory of elasticity can be expressed in terms of one single Westergaard function $Z(z)$, as derived by Sih (4)

$$\sigma_{xx} + \sigma_{yy} = Z(z) + \overline{Z(z)}$$

$$\sigma_{yy} - \sigma_{xx} + 2i\sigma_{xy} = 2A - (z - \bar{z})Z'(z) \tag{4}$$

$$2\mu(u + iv) = \frac{1}{2}\left[\frac{3-\nu}{1+\nu}\int Z(z)dz - \int \overline{Z(z)}d\bar{z} - (z - \bar{z})\overline{Z(z)}\right] - A\bar{z}$$

where A is a real constant, μ is the shear modulus and ν is Poisson's ratio.

In our further investigation we consider two yield criteria, the Tresca criterion and the Mises criterion. Since the assumed yield model depends principally on the evolution of plastic zone along the x-axis the two yield criteria can be written in terms of principal stresses as follows

$$\left[(\sigma_{xx} - \sigma_{yy})^2 - \sigma_0^2\right]\left[\sigma_{xx}^2 - \sigma_0^2\right]\left[\sigma_{yy}^2 - \sigma_0^2\right] = 0 \tag{5}$$

for the Tresca yield criterion, and

$$\sigma_{xx}^2 - \sigma_{xx}\sigma_{yy} + \sigma_{yy}^2 = \sigma_0^2 \tag{6}$$

for the Mises yield criterion, respectively.

The magnitude of the constant cohesive stress $\sigma_{yy} = \sigma_Y$ to be used in our computations is to be consistent with the actual stress state along the x-axis and any of two yield criteria considered. First, consistency of stresses along the x-axis implies that the difference of the normal stresses is constant and proportional to the applied loading at infinity

$$\sigma_{yy}(z) - \sigma_{xx}(z) = 2A = (1 - k)\sigma_\infty \qquad \textit{for} \quad Im\, z = 0 \tag{7}$$

and second, while obeying (7) and introducing a coefficient α according to

$$\alpha = \frac{1-k}{2}\frac{\sigma_\infty}{\sigma_0} \tag{8}$$

it can be demonstrated, Štok (5), that consistency of the crack tip zone yielding is proper only if

$$\sigma_Y^{Tresca} = \begin{cases} \sigma_0(1+2\alpha) & \ldots k > 1 \\ \sigma_0 & \ldots k \leq 1 \end{cases} \quad ; \quad \sigma_Y^{Mises} = \sigma_0(\alpha + \sqrt{1-3\alpha^2}) \tag{9}$$

is taken for the cohesive stress σ_Y. The range of admissible values for the coefficient α is obtained by considering a stable solution with the plastic zone localized at the crack tips and positive cohesive stress σ_Y. This results in

$$-0,5 < \alpha^{Tresca} < 0,5 \quad ; \quad -0,5 < \alpha^{Mises} < \frac{1}{\sqrt{3}} \tag{10}$$

Regarding the role of the coefficient α in the evolution of plastic yielding it is important to emphasize that while the biaxial load ratio k is assumed fixed for a considered loading case the coefficient α is subject to variation from zero to a maximum value, accordingly to the gradual application of the remote loading σ_∞. This fact has a tremendous impact on the cohesive stress behaviour. A thorough analysis deduced by Štok (5) shows that the cohesive stress σ_Y exhibits some kind of "softening" and "hardening" effects that are crucial for the nature of the actual plastic response.

PROBLEM SOLUTION AND DISCUSSION OF RESULTS

In accordance with Dugdale's approach the solution of the considered elastic-plastic boundary value problem can be obtained by splitting the problem in two elastic subproblems, one related to the application of the remote loading and the other related to the application of the loading due to the cohesive forces. Superposition of the subproblem solutions, determined by the correspondent Westergaard functions $Z_1(z)$ and $Z_2(z)$ of the form

$$Z_1(z) = \frac{z}{\sqrt{z^2-b^2}}\sigma_{yy}^{\infty} + \frac{1}{2}(\sigma_{xx}^{\infty} - \sigma_{yy}^{\infty}) \tag{11}$$

$$Z_2(z) = -\frac{2\sigma_Y}{\pi}\left[\frac{z}{\sqrt{z^2-b^2}}\arccos\left(\frac{a}{b}\right) - \arctan\left(\frac{z}{a}\sqrt{\frac{b^2-a^2}{z^2-b^2}}\right)\right] \tag{12}$$

yields after the removal of the resulted stress singularities at the crack tip according to the consistency condition (1), the problem solution in terms of the Westergaard function $Z(z)$

$$Z(z) = \frac{2\sigma_Y}{\pi} \arctan\left(\frac{z}{a}\sqrt{\frac{b^2 - a^2}{z^2 - b^2}}\right) + \frac{1}{2}(\sigma_{xx}^{\infty} - \sigma_{yy}^{\infty}) \tag{13}$$

with the half length b of the imaginary crack being specified by the relationship

$$\frac{a}{b} = \cos\left(\frac{\pi}{2}\frac{\sigma_{yy}^{\infty}}{\sigma_Y}\right) \quad . \tag{14}$$

Stresses and displacements in the domain D can be readily determined by considering relationships (4), while in order to characterize the fracture behaviour it is convenient to determine the crack tip opening displacement δ_t and the plastic extension ahead of the crack tip r_p. Both parameters can be expressed explicitly in terms of the applied remote loading σ_∞ and the correspondent cohesive stress σ_Y

$$\frac{\delta_t}{a} = \frac{8}{\pi}\frac{\sigma_Y}{E}\ln\frac{b}{a} = \frac{8}{\pi}\frac{\sigma_Y}{E}\ln\left[\sec\left(\frac{\pi}{2}\frac{\sigma_{yy}^{\infty}}{\sigma_Y}\right)\right] \tag{15}$$

$$\frac{r_p}{a} = \frac{b}{a} - 1 = \sec\left(\frac{\pi}{2}\frac{\sigma_{yy}^{\infty}}{\sigma_Y}\right) - 1 \quad . \tag{16}$$

At this point some conclusions can begiven regarding a role that the load biaxiality has on the plastic yielding. In general, larger the absolute value of k larger the sensitivity of the plastic response. Also, with respect to the same absolute value the negative values of k are more favourable and yield smaller plastic yielding. However, in order to thoroughly characterize and trace the plastic response, as the load is monotonically applied, one needs to consider the evolution of parameters affecting this response. The simplest case is undoubtedly the one studied by Dugdale (2) where the uniform load biaxiality ($k = 1$) is assumed. Because of $\alpha = 0$ the cohesive stress is constant $\sigma_Y = \sigma_o$ through the whole loading history and results, obtained by taking the considered yield criteria into account, do not differ. For $k > 1$ and $\alpha < 0$, however, a monotonic increase of the load is characterized by a monotonic decrease of the cohesive stress σ_Y. Due to this "softening" effect the rate of the plastic zone propagation increases progressively, irrespective of the yield criterion used. On the contrary, for $k < 1$ and $\alpha > 0$ the responses regulated by the two yield criteria differ substantially in their nature. With the Tresca yield criterion assumed the cohesive stress remains constant $\sigma_Y = \sigma_o$ through the whole loading history, thus exhibiting a "perfect plasticity" effect. Much more complex is the response if the Mises yield criterion is assumed. At low levels of the applied load the response is first characterized by a "hardening" effect and a monotonic increase of the cohesive stress σ_Y, as the load is monotonically increased. The rate of the plastic zone propagation is therefore rather moderate. While for $0{,}5 < k < 1$ "hardening" is characteristic for the plastic zone propagation irrespective of the applied load, this propagation is characterized for values $k < 0{,}5$ by a sudden "softening" after a certain level of the applied load is passed. When this occurs the rate of the plastic zone propagation increases progressively.

SYMBOLS USED

a	= physical crack half-length (mm)
b	= imaginary crack half-length (mm)
δ_t	= crack tip opening displacement (mm)
k	= biaxial load ratio
K	= stress intensity factor ($MPa\,mm^{1/2}$)
μ	= shear modulus (MPa)
ν	= Poisson's ratio
r_p	= plastic zone length (mm)
σ_{ij}	= components of stress tensor (MPa); $i,j \in \{x,y,z\}$
σ_0	= yield stress (MPa)
σ_Y	= cohesive stress (MPa)
u	= x-component of displacement vector (mm)
v	= y-component of displacement vector (mm)
z	= complex variable ($z = x + i.y$)
Z(z)	= Westergaard function

REFERENCES

(1) Rice, J.R., "Mathematical Analysis in the Mechanics of Fracture", Fracture - An Advanced Treatise, Edited by H. Liebowitz, Academic Press, New York and London, 1968.

(2) Dugdale, D.S., "Yielding of steel sheets containing slits", J. Mech. Phys. Solids, Vol. 8, 1960, pp. 100-104.

(3) Muskhelishvili, N.I., Some Basic Problems of the Mathematical Theory of Elasticity. P. Noordhoff Ltd, Groningen - The Netherlands, 1963.

(4) Sih, G.C., "On the Westergaard method of crack analysis", J. Fracture Mech.,Vol. 2, 1966, pp. 628-630.

(5) Štok, B., "Reconsidering of some issues related to generalization of Dugdale's approach to the crack tip plasticity", submitted for publication in Engng Fracture Mech.