CRACK KINKING IN ANISOTROPIC MATERIALS

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The stress singularity at the apex of a kinked crack in an anisotropic material is studied. The starting point is the solution of Stroh (1) for a dislocation in an anisotropic material. The crack and the branch are modelled as continuous distributions of such dislocations, which are assumed to be singular at the apex, the kind of singularity being unknown and weaker than at the crack tip. The Mellin transform is used to obtain a system of simultaneous functional equations that permits to find the kind of singularity. Results are presented to compare the present analysis with existing solutions for some particular geometries and material models such as isotropic sharp angular notches. For the general anisotropic case, results are presented showing the influence of geometric and material parameters.

INTRODUCTION

The elastic stress singularity in a neighbourhood of the apex of a kinked crack in an anisotropic material is analysed. The state is three-dimensional but the stress and displacement fields are assumed to be independent of the x_3 coordinate $-u_k = u_k(x_1, x_2)-$. Stroh (1) obtained the solution for a dislocation in an infinite homogeneous medium with general anisotropy.

In section 2, the kinked crack is modelled as a continuous distribution of such dislocations. A system of simultaneous functional equations is obtained applying the Mellin transform. The analysis of the analyticity of these equations permits to find the kind of singularity of the stresses at the apex.

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In section 3 numerical results obtained by means of this analysis are compared with existing solutions for some particular geometries (i.e., sharp angular notches) in which the behaviour of the stresses is analogous. Results for kinked cracks in media with general anisotropy are presented.

ASYMPTOTIC ANALYSIS OF THE SINGULAR STRESS FIELD AT THE APEX OF A KINKED CRACK

Statement of the problem

Consider a coordinate system (O, x, y) such that the main crack lies in the plane y = 0, the apex is at the origin and the branch line makes an angle ϕ with the x-axis (Fig. 1). In a neighbourhood of the apex the stress field is singular, but the singularity is weaker than at the crack tip, namely, verify (Bogy (2))

$$\sigma_{ij}(r,\theta) \approx r^{-\lambda} \qquad (r \to 0, \ -\pi < \theta < \phi) \quad \ldots\ldots(1)$$

with $0 < \lambda < 1/2$, and where (r, θ) denote the polar coordinates.

The objective is, therefore, to find λ for a kinked crack in an anisotropic material, with angle ϕ where the lengths of the main crack and of the branch are arbitrary.

Since we consider a neighbourhood of the apex, the lengths of the main crack and of the branch do not affect the kind of singularity of the stresses. Therefore, both the main crack and the branch will be assumed to be semi-infinite.

Mathematical formulation

A new coordinate system (O, x', y') is introduced so that the branch line corresponds to the x'-axis (Fig. 2). Now we consider an infinite crack with two semi-infinite branches. In order to simplify the notation, the terms *main crack* and *branch* will be used to denote the negative Ox semi-axis and the positive Ox' semi-axis, respectively.

The solution for a dislocation in an infinite anisotropic medium is (Stroh (1))

$$\sigma_{i1}(x,y) = \frac{-1}{4\pi}\sum_{\alpha} L_{i\alpha} M_{\alpha j} p_{\alpha} \frac{d_j}{z_{\alpha} - \zeta_{\alpha}} + C.C.$$
$$\ldots\ldots(2)$$
$$\sigma_{i2}(x,y) = \frac{1}{4\pi}\sum_{\alpha} L_{i\alpha} M_{\alpha j} \frac{d_j}{z_{\alpha} - \zeta_{\alpha}} + C.C.$$

where C.C. denotes the complex conjugate of the preceding expression and the convention of summing over repeated indices is used. The magnitudes p_{α}, $L_{i\alpha}$, $M_{\alpha j}$, z_{α}, ζ_{α} and d_j are designated with the same name in Stroh's work.

The crack is modelled as a continuous distribution of dislocations, that we will designate f_j and g_j if they are on the main crack or on the branch respectively. These functions are expected to be singular at the origin; therefore, we can express

$$f_j(\zeta) = \frac{F_j(\zeta)}{(-\zeta)^{\lambda}} \quad \text{and} \quad g_j(\eta) = \frac{G_j(\eta)}{\eta^{\lambda}} \quad \text{.......(3)}$$

where $F(\zeta)$ and $G(\eta)$ are bounded functions in (-∞,0) and (0,∞), respectively.

To determine the distribution of dislocations use is made of the condition that the crack is traction free, therefore,

$$-t_i(x) = \frac{1}{4\pi}\sum_{\alpha} L_{i\alpha}M_{\alpha j}\left\{\int_{-\infty}^{0}\frac{f_j(\zeta)}{x-\zeta}d\zeta + \int_{0}^{\infty}\frac{g_j(\eta)}{x-\eta\tau_\alpha}d\eta\right\} + C.C. \quad (x<0) \quad \text{.......(4.a)}$$

$$-t'_k(x') = \frac{l_{ki}}{4\pi}\sum_{\alpha} L_{i\alpha}M_{\alpha j}\tau_\alpha\left\{\int_{-\infty}^{0}\frac{f_j(\zeta)}{x'\tau_\alpha-\zeta}d\zeta + \frac{1}{\tau_\alpha}\int_{0}^{\infty}\frac{g_j(\eta)}{x'-\eta}d\eta\right\} + C.C. \quad (x'>0)$$

(4.b)

where

$$\tau_\alpha = \cos\phi + p_\alpha \sin\phi \quad \text{.......(5)}$$

$$t_i(x) = t_{i2}(x,0) \quad \text{and} \quad t'_k(x') = t'_{k2}(x',0) \quad \text{.......(6)}$$

t_{ij} and t'_{kj} being the stress tensors in the medium without crack referred to the coordinate systems (O, x, y) and (O, x', y') respectively. l_{ki} is the matrix of the change of coordinates.

Applying some changes of variable, the Mellin transform and doing the algebra, equations (4.a,b) turn into

$$\tilde{t}_i^*(s) = \frac{1}{4\pi}\sum_{\alpha} L_{i\alpha}M_{\alpha j}\left\{-\int_0^\infty f_j^*(\zeta)\zeta^{s-1}d\zeta\int_0^\infty\frac{r^{s-1}}{r-1}dr - \int_0^\infty g_j(\eta)\eta^{s-1}d\eta\int_0^\infty\frac{r^{s-1}}{r+\tau_\alpha}dr\right\}$$

$$+\frac{1}{4\pi}\sum_{\alpha}\bar{L}_{i\alpha}\bar{M}_{\alpha j}\left\{-\int_0^\infty f_j^*(\zeta)\zeta^{s-1}d\zeta\int_0^\infty\frac{r^{s-1}}{r-1}dr - \int_0^\infty g_j(\eta)\eta^{s-1}d\eta\int_0^\infty\frac{r^{s-1}}{r+\bar{\tau}_\alpha}dr\right\} \quad \text{..(7.a)}$$

$$\tilde{t}_k'^*(s) = \frac{l_{ki}}{4\pi}\sum_{\alpha} L_{i\alpha}M_{\alpha j}\tau_\alpha\left\{\int_0^\infty f_j^*(\zeta)\zeta^{s-1}d\zeta\int_0^\infty\frac{r^{s-1}}{r\tau_\alpha+1}dr + \frac{1}{\tau_\alpha}\int_0^\infty g_j(\eta)\eta^{s-1}d\eta\int_0^\infty\frac{r^{s-1}}{r-1}dr\right\}$$

$$+\frac{l_{ki}}{4\pi}\sum_{\alpha}\bar{L}_{i\alpha}\bar{M}_{\alpha j}\bar{\tau}_\alpha\left\{\int_0^\infty f_j^*(\zeta)\zeta^{s-1}d\zeta\int_0^\infty\frac{r^{s-1}}{r\bar{\tau}_\alpha+1}dr + \frac{1}{\bar{\tau}_\alpha}\int_0^\infty g_j(\eta)\eta^{s-1}d\eta\int_0^\infty\frac{r^{s-1}}{r-1}dr\right\}$$

(7.b)

with

$$f_j^*(\zeta) = f_j(-\zeta) \quad \text{.......(8)}$$

$$t_i^*(x_1) = -t_i(-x_1) \quad \text{and} \quad t_k'^*(x') = -t'_k(x') \quad \text{.......(9)}$$

where $\tilde{t}_i^*(s)$ and $\tilde{t}_k'^*(s)$ are the Mellin transforms of the functions $t_i^*(x_1)$ and $t_k'^*(x')$, respectively.

Defining

$$\hat{f}_j(s)=\int_0^\infty f_j^*(\zeta)\zeta^{s-1}d\zeta \qquad \text{and} \qquad \hat{g}_j(s)=\int_0^\infty g_j(\eta)\eta^{s-1}d\eta \qquad (10)$$

equations (7.a,b) can be written in matrix form as

$$\left(\begin{array}{c|c} \frac{\cot(\pi s)}{2}\mathbf{I}_{3\times3} & -\frac{\csc(\pi s)}{4}H_{ij}(s-1) \\ \hline l_{ki}\frac{\csc(\pi s)}{4}H_{ij}(1-s) & -l_{ki}\frac{\cot(\pi s)}{2} \end{array}\right)\cdot\begin{pmatrix}\hat{f}_i(s)\\ \hat{g}_j(s)\end{pmatrix}=\begin{pmatrix}\tilde{t}_i^*(s)\\ \tilde{t}_k'^*(s)\end{pmatrix} \qquad (11)$$

with

$$H_{ij}(z)=\sum_\alpha L_{i\alpha}M_{\alpha j}\tau_\alpha^z+\sum_\alpha \bar{L}_{i\alpha}\bar{M}_{\alpha j}\bar{\tau}_\alpha^z \qquad (12)$$

and $\mathbf{I}_{3\times3}$ is the identity matrix.

Since $H_{ij}(\zeta)$ are entire functions, the elements of the system matrix are analytic at the domain $0<\mathrm{Re}(s)<1$. On the other hand, using equations (3), (8) and (10), the functions $\hat{f}_j(s)$ can be written as

$$\hat{f}_j(s)=\int_0^\infty \frac{F_j(-\zeta)}{\zeta^\lambda}\zeta^{s-1}\,d\zeta, \qquad (13)$$

and therefore, they are analytic in the half-plane $\lambda<\mathrm{Re}(s)$. Likewise the functions $\hat{g}_j(s)$ are analytic in the same domain. Finally, the elements of the matrix of independent terms are analytic in the half-plane $0<\mathrm{Re}(s)$ since both functions $t_i^*(x)$ and $t_k'^*(x')$ are regular. From these observations we can conclude that the inverse of the system matrix $\mathbf{M}^{-1}(s)$ must be singular at a point s_0 with $\mathrm{Re}(s_0)=\lambda$, that is, the equation

$$\det\,[\mathbf{M}(s)]=0 \qquad (14)$$

has a root whose real part is the parameter λ that characterises the kind of singularity of the stresses in a neighbourhood of the apex.

Using the Cramer's rule and the Mellin transform inversion theorem we obtain the unknown functions $\hat{f}_j(s)$ and $\hat{g}_j(s)$, and applying the residue theorem we can conclude that $g_j(\zeta)\approx\zeta^{-s_0}$ when $\zeta\to 0$, s_0 being the singularity of greatest real part and $\mathrm{Re}(s_0)>0$.

For the solution to have physical sense, the root s_0 should be real. This property has not been proved analytically, but it is verified in the numerical results obtained.

RESULTS

Table 1 shows the results obtained using the present formulation for different materials and angles between the main crack and the branch.

The stress and displacement fields in a neighbourhood of the apex of a kinked crack in an isotropic material are known (Atkinson et al (4) and Sih and Ho (5)). In this case, the kind of singularity does not depend on the elastic constants. In this table the results contained in (4) for a sharp angular notch are compared with the values obtained with our analysis approximating the isotropic material as a limit of slight anisotropy.

The material M-1 is di-potassium tartrate and M-2 is sodium thiosulfate; both are monoclinic materials. The anisotropic materials A-1 and A-2 are not real materials. Positive-definite, symmetric matrices with arbitrary elements were taken to define their elastic constants. It can be seen that λ varies slightly for different anisotropic materials.

TABLE 1 - Values of λ for anisotropic and isotropic materials.

angle ϕ	M-1	M-2	A-1	A-2	slight anisotropy	Ref (4)
0°	0,000000	0,000000	0,000000	0,000000	0,000000	0,000000
20°	0,178866	0,163532	0,183998	0,240036	0,181304	0,181304
40°	0,311446	0,297122	0,312419	0,368805	0,302835	0,302835
60°	0,397000	0,393772	0,398110	0,427215	0,384269	0,384269
80°	0,443818	0,445523	0,444589	0,454589	0,437161	0,437161
100°	0,470275	0,472203	0,470592	0,470955	0,469604	0,469604
120°	0,486573	0,487943	0,486679	0,482904	0,487779	0,487779
140°	0,495938	0,496651	0,496014	0,492472	0,496510	0,496510
160°	0,499516	0,499631	0,499535	0,498691	0,499574	0,499574
180°	0,500000	0,500000	0,500000	0,500000	0,500000	0,500000

As an internal check, note that for $\phi = 0°$, the main crack and the branch are on the same plane; the value $\lambda = 0$ is then consistent with the absence of singularity as there is no kink in this case. The other limit case is when $\phi = 180°$, for which the value $\lambda = 1/2$ agrees with the known stress singularity at the crack tip.

CONCLUSIONS

We have developed an analytic-numerical method in order to obtain the kind of singularity of the stresses in a neighbourhood of the apex of a kinked crack in materials with general anisotropy. The validity of the method described in this paper has been checked against the solution for an angular notch in an isotropic material, which has been analysed as a limit case of our formulation. For the general anisotropic problem, results have been presented to show the influence of geometric and material parameters on the kind of singularity.

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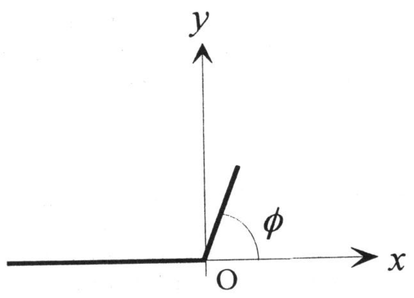

Figure 1 Kinked crack.

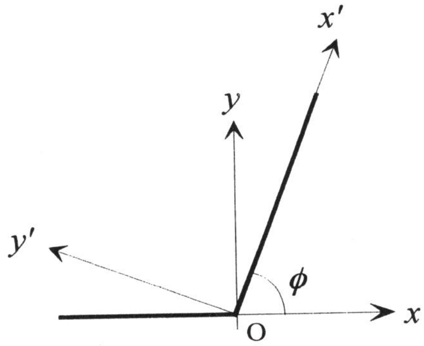

Figure 2 Coordinate system.