BOUNDARY INTEGRO-DIFFERENTIAL EQUATIONS FOR ANISOTROPIC CRACK PROBLEMS

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The boundary formulation for crack problems in a fully anisotropic medium is investigated. First, the so-called limit theorems describing the limit behaviour of the fundamental solution near a closed or open surface are given. These theorems are generalization of the well-known ones in the isotropic case.
Next, the boundary integro-differential equations are derived for the problem of anisotropic cracked bodies. The formulation includes both the cases of the infinite body (with an embedded crack) and a finite body with an embedded or surface crack.
Throughout the paper, emphasis is made on the mathematical conditions for the results to be valid.

INTRODUCTION

The boundary integral equation method for an anisotropic elastic three-dimensional continuum was first investigated by Vogel and Rizzo (1), followed by an efficient numerical implementation proposed by Wilson and Cruse (2). In the field of anisotropic fracture mechanics, the boundary formulation was discussed in the pioneering works of Sladek and Sladek (3), Balas et al (4).

This paper deals with the boundary formulation for crack problems in a fully anisotropic medium as a continuation of the previous works. First, the so-called limit theorems describing the limit behaviour of the fundamental solution near a closed or open surface are given. These results are generalization of the well-known ones in the isotropic case. Their proofs require a minimum amount of basic

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properties only and not the knowledge of the closed form of the fundamental solution. Next, the boundary integral equations are derived for both the cases of the infinite body (with an embedded crack) and a finite body with an embedded or surface crack.

THE LIMIT THEOREMS

The purpose of this section is to present some results about the limit behaviour of the fundamental solution when the load point x approaches a point y_0 belonging to a given surface S. Consider an isotropic body characterized by the fourth-order elastic tensor C whose components written in a fixed base ($e_1e_2e_3$) verify the usual symmetries : $C_{ijk\ell}=C_{jik\ell}=C_{ij\ell k}=C_{k\ell ij}$. Let us introduce the following definition where the summation convention is implied over repeated subscripts which all have the range (1,2,3).

Definition

- Given a unit vector e_m of the base ($e_1e_2e_3$), the fundamental solution related to e_m and denoted by $U(e_m,x,y)$ is the solution of the partial differential equation in the infinite three-dimensional space $\mathcal{E}$:

$\mathrm{div}\,[C : \mathrm{grad}\,U(e_m,x,y)] + \delta(y-x)e_m = 0$ where $\delta(y\text{-}x)$ is the Dirac function.

- The corresponding stress tensor is defined as :

$$\Sigma(e_m,x,y) = t_{U(e_m,x,y)}(y,n_y = e_j) \otimes e_j = C : \mathrm{grad}U(e_m,x,y)$$

where the tensor product $a\otimes b$ is defined by $(a\otimes b)_{ij}=a_ib_j$. The differentiation is performed with respect to variable y and $t_{U(e_m,x,y)}(y,n_y)$ denotes the stress vector at point y with respect to normal n_y and corresponding to the displacement field $U(e_m,x,y)$.

- Now the fundamental displacement tensor is defined by :

$$E(x,y) = U(e_m,x,y) \otimes e_m, \text{ i.e. } E_{ij}(x,y) = U_i(e_j,x,y)$$

- This, in turn, gives rise to the third-order tensor of the fundamental stress :

$$D(x,y) = \Sigma(e_m,x,y) \otimes e_m, \text{ i.e. } D_{ijk}(x,y) = \Sigma_{ij}(e_k,x,y) = C_{ijpq}\frac{\partial E_{pk}}{\partial y_q}(x,y)$$

- Eventually, the Kupraze tensor is defined by (See reference 5) :

$$T(x,y,n_y) = t_{U(e_m,x,y)}(y,n_y) \otimes e_m, \text{ i.e. } T_{ik}(x,y,n_y) = C_{ijpq}n_j(y)\frac{\partial E_{pk}}{\partial y_q}(x,y)$$

We have : $T_{ik}(x,y,n_y) = \Sigma_{ij}(e_k,x,y)n_j(y) = D_{ijk}(x,y)n_j(y)$.

Conversely : $\Sigma_{ij}(e_k,x,y) = D_{ijk}(x,y) = T_{ik}(x,y,n_y= e_j)$.

As tensors Σ and T are functions of D, any equation in the sequel can be expressed in terms of D alone. In practice however, the simultaneous use of notations T and D proves to be more convenient.

The integral representation of the fundamental solution $U(e_m,x,y)$ for an isotropic elastic medium was given in (1) by decomposing the Dirac function into plane waves. Although the closed form solution is not available, various basic properties on the asymptotic behaviour of tensors $\partial E/\partial y$, T and D can be deduced from the integral representation, which make it possible to prove the so-called limit theorems below.

Before stating the limit theorems, let us first agree about notations for the orientation of a surface. Of course, any surface S (closed or open) considered here is assumed to be orientable. This implies that, for any point $y_0 \in S\backslash\partial S$, we can locally define two sides of S which we label side + and side -, all the normals to S being directed from side - to side +.

Theorem 1

Let S be a surface (closed or open) and u a vector field defined on S. If :

i) S is a Lyapunov surface : $S \in C^{1,\alpha}$, $0<\alpha\leq 1$

ii) u satisfies the Holder condition on S : $u \in C^{0,\beta}(S)$

, then :

$$\forall y_0 \in S, \quad \lim_{x \to y_o^{\pm}} \int_S T(x,y,n_y)u(y)\, d_yS = \pm\frac{1}{2}u(y_o) + pv\int_S T(y_o,y,n_y)u(y)\, d_yS$$

where by $x \to y_o^{\pm}$ are meant the limits as x approaches y_0, x belonging to the side + and the side - of S, respectively. The symbol pv denotes a Cauchy principal value integral.

The following theorem requires a somewhat stronger condition for the function u.

Theorem 2

Assuming that :

i) $S \in C^{1,\alpha}$, $0<\alpha\leq 1$

ii) $u \in C^{1,\beta}(S)$ $0<\beta\leq 1$, i.e. all the derivatives of u belong to the class $C^{0,\beta}(S)$

, we have the property of continuity across the boundary :

$$\forall y_0 \in S, \quad \lim_{x \to y_o^\pm} \int_S \mathcal{R}(\partial y,x,y,n_y)u(y)\, d_yS\, .n_{yo} = pv \int_S \mathcal{R}(\partial y,y_o,y,n_y)u(y)\, d_yS.n_{yo}$$

where the symbol $\mathcal{R}$ represents the differential operator defined as (4) :

$$(\mathcal{R}(\partial y,x,y,n_y)u(y))_{ij} = \mathcal{R}_{ijm}(\partial y,x,y,n_y)u_m(y) = C_{ijk\ell}D_{mnk}(x,y).\mathcal{D}_{n\ell}(\partial y,n_y)u_m(y)$$

,$\mathcal{D}_{n\ell}(\partial y,n_y)$ is the tangential differential operator defined as :

$$\mathcal{D}_{n\ell}(\partial y,n_y) = n_n(y)\frac{\partial}{\partial y_\ell} - n_\ell(y)\frac{\partial}{\partial y_n}$$

Here the hypothesis for the displacement, $u \in C^{1,\beta}(S)$, is stronger than that in theorem 1 because of the derivatives involved by the differential operator $\mathcal{R}$.

The above relations, established in the anisotropic case, constitute the generalization of well-known results in isotropy where they can be directly verified using the closed form expressions available in (4) for T, D and $\mathcal{R}$.

BOUNDARY INTEGRO-DIFERENTIAL EQUATION (BIDE)

This section gives the boundary integro-diferential equation (BIDE) for the problem of anisotropic cracked bodies. Two cases are considered : the infinite body (with an embedded crack) and a finite body with an embedded or surface crack.

Consider a linear elastic anisotropic, finite or infinite, body Ω containing a crack. If the body is finite, its outer boundary is denoted S_B and the crack can be either an embedded one or a surface one. The crack surface S_{cr} is made up of two faces S_{cr}^+ and S_{cr}^- which coincide in the undeformed state. To each point $y \in S_{cr}$ correspond two points y^+ and y^- belonging respectively to S_{cr}^+ and S_{cr}^-. The respective normal vectors are opposite, i.e. $n_y^+ = n_y^-$, n_y^- being directed from S_{cr}^- to S_{cr}^+, thus defined everywhere as outward with respect to the body considered, in accordance with the usual convention. In the sequel, all the equations will be written using S_{cr}^-, so that normal n_y^- is taken as the reference one.

Theorem 3

Consider the infinite body Ω containing the crack $S_{cr} = S_{cr}^- \cup S_{cr}^+$. If :

i) the regularity conditions are fulfilled : $u(y)=o(1)$, $\sigma(y)=o(1/r)$ i.e. $t(y,n_y)=o(1/r)$ when $r=\|y-x\| \to \infty$, x being a fixed point; and $f(y)=O(r^{-2-\delta})$ when $r \to \infty$, $0<\delta\leq 1$

ii) $S_{cr}^- \in C^{1,\alpha}$, $0 < \alpha \leq 1$

iii) the displacement jump through the crack $\Delta u(y)=u(y^+)-u(y^-)$ verifies : $\Delta u \in C^{1,\beta}(S_{cr}^-)$, $0<\beta\leq 1$. (From hypotheses (ii) and (iii), it follows that the sum of the stress vectors on the crack faces $\Sigma t(y)=t(y^+,n_y^+)+t(y^-,n_y^-)$ verifies : $\Sigma t = -[C:(\mathrm{grad}\Delta u)].n_y^- \in C^{0,\beta'}(S_{cr}^-)$, $0<\beta'\leq 1$)

, then the boundary integro-differential equation writes :

$$\forall y_o \in S_{cr},\ \frac{1}{2}[\ t(y_o^-,n_{yo}^-) - t(y_o^+,n_{yo}^+)] = *\int_\Omega D(y,y_o)f(y)\,d_yV.\,n_{yo}^-$$

$$+ \mathrm{pv}\int_{S_{cr}^-} \{\ D(y,y_o)\Sigma t(y) + \mathcal{R}(\partial y,y_o,y,n_y)\Delta u(y)\ \}\,d_yS.\,n_{yo}^-$$

where the asterisk denotes an improper integral.

If the crack is loaded symmetrically, i.e. $t(y^+,n_y^+)=-t(y^-,n_y^-)$, then $\Sigma t(y)=0$. If the loads t^+ and t^- are specified otherwise, the solution of the above BIDE gives the displacement jump Δu, and then u^+ and u^- using the integral representation of the displacement. Conversely, if Δu is prescribed, the BIDE does not allow to obtain t^+ and t^- separately, unless an additional information is supplied, e.g. $t^+=-t^-$.

Theorem 4

Consider a finite body Ω with outer boundary S_B, containing an embedded or surface crack S_{cr}. Assuming that :

i) $S_{cr}^- \in C^{1,\alpha}$, $S_B \in C^{1,\alpha'}$, $0<\alpha,\alpha'\leq 1$

ii) $\Delta u \in C^{1,\beta}(S_{cr}^-)$, $u \in C^1(S_B)$

, we have the following system of boundary integr(o-differenti)al equations :

$$\forall y_o\in S_B,\ \int_{S_{cr}^-} [\ E(y_o,y)\Sigma t(y)+T^T(y_o,y,n_y)\Delta u(y)\ ]d_yS\ +$$

$$*\int_{S_B} \{\ E(y_o,y)t(y,n_y)-T^T(y_o,y,n_y)[u(y)-u(y_o)]\ \}d_yS + *\int_\Omega E(y_o,y)f(y)d_yV = 0$$

and :

$$\forall y_o\in S_{cr},\frac{1}{2}[t(y_o^-,n_{yo}^-)-t(y_o^+,n_{yo}^+)] = \int_{S_B} \{D(y,y_o)t(y,n_y)-\mathcal{R}(\partial y,y_o,y,n_y)u(y)\}d_yS.n_{yo}^-$$

$$+\mathrm{pv}\int_{S_{cr}^-} \{D(y,y_o)\Sigma t(y)+\mathcal{R}(\partial y,y_o,y,n_y)\Delta u(y)\}d_yS.n_{yo}^- + *\int_\Omega D(y,y_o)f(y)\,d_yV.\,n_{yo}^-$$

In the case of a surface crack, S_B must be replaced by $S_B \backslash L$ throughout the above equation system, where $L=\partial S_{cr}^{-} \cap S_B$ is referred to as the surface line.

In the case of a surface crack, it has been assumed that $u \in C^1(S_B \backslash L)$, not $u \in C^1(S_B)$. Indeed, the surface line L corresponding to an incision in the boundary S_B, gives rise to a *displacement discontinuity* on S_B along L and makes the hypothesis $u \in C^1(S_B)$ impossible. The theorem 4 cannot be proved by considering $S_B \cup S_{cr}$ as a single closed surface. The crack geometry clearly indicates that , contrary to S_{cr}^{-}, $S_{cr}=S_{cr}^{+} \cup S_{cr}^{-}$ cannot be assumed to belong to the class $C^{1,\alpha}$.

The solution of the above system give (u,t) on S_B and $(\Delta u, \Sigma t)$ on S_{cr}. The coupling between these equations accounts for the interaction between the outer boundary and the crack. In the case of a surface crack, the system provides no equations along the surface line, since $y_o \notin L$. However, the lacking equations are compensated for by expressing the displacement compatibility at the surface line. With obvious notations, we can write at every point on L :

$$\Delta u(y_o \in L) + u(y_o^- \in L^-) - u(y_o^+ \in L^+) = 0$$

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