ANALYTICAL SOLUTION OF THREE-DIMENSIONAL PROBLEM FOR ELLIPTICAL CRACK SUBJECTED TO ARBITRARY TIME-HARMONIC LOADS

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The problems for elliptical cracks subjected to arbitrary time-harmonic loads are considered. Traction boundary pseudodifferential equations are used for the solution to the problems. The constructive analytical procedure is developed for the expansion of the equations solutions into the Taylor's series by the wave number. The solutions are not analytical functions of the wave number therefore the Taylor's series converge only for low wave numbers. Pade approximants are used to extend the range of accurate approximation of the solution from low to intermediate wave numbers.

STATEMENT OF THE PROBLEM

Let the crack occupies the region G in the plane $x_3 = 0$ of an infinite elastic solid. Assume that time-harmonic loads with amplitudes $\pm t = \pm(t_1, t_2, t_3)$, $t_i = t_i(\beta, x)$ are applied to the crack surfaces. Here $x = (x_1, x_2) \in G$, β is the wave number ($\beta = \omega/C_s$, ω is the angular frequency, C_s is the transverse speed). It is presumed that any time-harmonic motions are superposed upon a statically open crack, achived say by applying tension at infinity in the x_3 - direction, sufficient to always ensure a gap between the faces. The radiation conditions are assumed at the infinity. It is well known that the problem can be reduced to the following boundary equations

$$p_G K_{33}(\beta)[u_3] = t_3 \quad (1)$$

$$p_G K_{11}(\beta)[u_1] + p_G K_{12}(\beta)[u_2] = t_1 \quad (2)$$

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$$p_G K_{12}(\beta)[u_1] + p_G K_{22}(\beta)[u_2] = t_2$$

$$[u_i] \in \dot{H}_{1/2}(G), \ \ t_i \in H_{-1/2}(G),$$

where $[u_i] = [u_i(\beta, x)]$ are amplitudes of crack opening displacements, $K_{ij}(\beta)$ are pseudodifferential operators, p_G is the restriction operator to the crack domain, $\dot{H}_{1/2}(G)$ and $H_{-1/2}(G)$ are Sobolev spaces. The symbols of operators $K_{ij}(\beta)$ are the following

$$K_{33}(\beta,\xi) = \mu(2\beta^2)^{-1}\{4\xi^2(\xi^2-\beta^2)^{1/2} - (2\xi^2-\beta^2)^2(\xi^2-\eta^2\beta^2)^{-1/2}\}$$

$$K_{\delta\delta}(\beta,\xi) = -\mu(2\beta^2)^{-1}\{4\xi_\delta^2(\xi^2-\beta^2)^{1/2} - \beta^2[(\xi^2-\beta^2)^{1/2} -$$

$$-\xi_\delta^2(\xi^2-\beta^2)^{-1/2}] - 4\xi_\delta^2(\xi^2-\eta^2\beta^2)^{1/2}\}$$

$$K_{12}(\beta,\xi) = -\mu(2\beta^2)^{-1}\xi_1\xi_2\{4(\xi^2-\beta^2)^{1/2}+\beta^2(\xi^2-\beta^2)^{-1/2}-4(\xi^2-\eta^2\beta^2)^{1/2}\}$$

Here $\delta = 1, 2$, μ is the shear modulus, $\eta^2 = (1-2\nu)/(2(1-\nu))$, ν is the Poisson's ratio, $\xi = (\xi_1, \xi_2)$, $\xi^2 = \xi_1^2 + \xi_2^2$.

Assume that the crack region G is an ellipse $G = \{(x_1, x_2) : x_1^2/a_1^2 + x_2^2/a_2^2 \leq 1, \ 0 < a_2 \leq a_1\}$ and the loads amplitudes t_k can be expanded into the power series by the wave number β

$$t_k = \sum_{N=0}^{\infty} (\beta a_2)^N (P_{Nk}(y) + iQ_{Nk}(y)) \tag{3}$$

where $y = (y_1, y_2)$, $y_i = x_i/a_i$, i = 1, 2, $P_{Nk}(y)$ and $Q_{Nk}(y)$ are polynomials, deg $P_{Nk}(y) \leq N + J$, deg $Q_{Nk}(y) \leq N + J$, J is an arbitrary nonnegative integer.

In this case the analytical constructive procedure for the expansion of the solution to the equations (1), (2) into power series by the wave number is developed.

THE METHOD OF SOLUTION EXPANSION INTO THE TAYLOR'S SERIES BY THE WAVE NUMBER

Denote $\rho^2 = y_1^2 + y_2^2$; $\varphi_a^\gamma(x) = (1-\rho^2)^\gamma$, $\rho < 1$; $\varphi_a^\gamma(x) = 0$, $\rho \geq 1$; $T_{pq}^\gamma(x) = y_1^p y_2^q \varphi_a^\gamma(x)$.

The developed analytical method is based on the following result. Functions $p_G K_{lj}(\beta)T_{pq}^{1/2}(x)$ are calculated analytically and have the form

$$p_G K_{lj}(\beta)T_{pq}^{1/2}(x) = \sum_{n=0}^{\infty}(\beta a_2)^{2n} U_{2n}^{pqlj}(y) + i\sum_{n=0}^{\infty}(\beta a_2)^{2n+3} V_{2n+3}^{pqlj}(y) \qquad (4)$$

where $U_{2n}^{pqlj}(y)$, $V_{2n+3}^{pqlj}(y)$ are polynomials, deg $U_{2n}^{pqlj}(y) = 2n + p + q$, deg $V_{2n+3}^{pqlj}(y) = 2n$. The explicit form of functions $U_{2n}^{pqlj}(y), V_{2n+3}^{pqlj}(y)$ is too cumbersome and doesn't represent here. Note only that $V_3^{pq12}(y) = 0$.

In consequence of formulae (3), (4) the solutions to the equations (1), (2) can be found in the following forms

$$[u_j] = \sum_{m=0}^{\infty}(\beta a_2)^m \sum_{p+q\le m+J}(B_{pq}^{jmr} + iB_{pq}^{jmi})T_{pq}^{1/2}(x) \qquad (5)$$

where B_{pq}^{jmr}, B_{pq}^{jmi} are real constants.

Substituting (3), (5) in (1), (2) and using (4) one gets equalities between power series by the βa_2 . Equating the coefficients of these series one obtains the system of the equations

$$\sum_{p+q\le N+J} B_{pq}^{3Nr} U_0^{pq33}(y) = P_{N3}(y) - \sum_{n=1}^{[N/2]}\sum_{p+q\le N+J-2n} B_{pq}^{3,N-2n,r} U_{2n}^{pq33}(y) +$$

$$+W(N)\sum_{n=0}^{[(N-3)/2]}\sum_{p+q\le N+J-2n-3} B_{pq}^{3,N-2n-3,i} V_{2n+3}^{pq33}(y) \qquad (6)$$

$$\sum_{p+q\le N+J} B_{pq}^{3Ni} U_0^{pq33}(y) = Q_{N3}(y) - \sum_{n=1}^{[N/2]}\sum_{p+q\le N+J-2n} B_{pq}^{3,N-2n,i} U_{2n}^{pq33}(y) -$$

$$-W(N)\sum_{n=0}^{[(N-3)/2]}\sum_{p+q\le N+J-2n-3} B_{pq}^{3,N-2n-3,r} V_{2n+3}^{pq33}(y)$$

Here $W(N) = 0$, $N = 0, 1, 2$ and $W(N) = 1$, $N \ge 3$, square brackets denote the greatest integer in the value.

$$\sum_{p+q\le N+J}[B_{pq}^{1Nr}\begin{pmatrix}U_0^{pq11}(y)\\ U_0^{pq12}(y)\end{pmatrix} + B_{pq}^{2Nr}\begin{pmatrix}U_0^{pq12}(y)\\ U_0^{pq22}(y)\end{pmatrix}] = \begin{pmatrix}P_{N1}(y)\\ P_{N2}(y)\end{pmatrix} -$$

$$-\sum_{n=1}^{[N/2]}\sum_{p+q\le N+J-2n}[B_{pq}^{1,N-2n,r}\begin{pmatrix}U_{2n}^{pq11}(y)\\U_{2n}^{pq12}(y)\end{pmatrix}+B_{pq}^{2,N-2n,r}\begin{pmatrix}U_{2n}^{pq12}(y)\\U_{2n}^{pq22}(y)\end{pmatrix}]+$$

$$+\sum_{n=0}^{[(N-3)/2]}\sum_{p+q\le N+J-2n-3}[B_{pq}^{1,N-2n-3,i}\begin{pmatrix}V_{2n+3}^{pq11}(y)\\0\end{pmatrix}+B_{pq}^{2,N-2n-3,i}\times$$

$$\times\begin{pmatrix}0\\V_{2n+3}^{pq22}(y)\end{pmatrix}]+\sum_{n=0}^{[(N-5)/2]}\sum_{p+q\le N+J-2n-5}[B_{pq}^{1,N-2n-5,i}\begin{pmatrix}0\\V_{2n+5}^{pq12}(y)\end{pmatrix}+$$

$$+B_{pq}^{2,N-2n-5,i}\begin{pmatrix}V_{2n+5}^{pq12}(y)\\0\end{pmatrix}] \tag{7}$$

$$\sum_{p+q\le N+J}[B_{pq}^{1Ni}\begin{pmatrix}U_0^{pq11}(y)\\U_0^{pq12}(y)\end{pmatrix}+B_{pq}^{2Ni}\begin{pmatrix}U_0^{pq12}(y)\\U_0^{pq22}(y)\end{pmatrix}]=\begin{pmatrix}Q_{N1}(y)\\Q_{N2}(y)\end{pmatrix}-$$

$$-\sum_{n=1}^{[N/2]}\sum_{p+q\le N+J-2n}[B_{pq}^{1,N-2n,i}\begin{pmatrix}U_{2n}^{pq11}(y)\\U_{2n}^{pq12}(y)\end{pmatrix}+B_{pq}^{2,N-2n,i}\begin{pmatrix}U_{2n}^{pq12}(y)\\U_{2n}^{pq22}(y)\end{pmatrix}]-$$

$$-\sum_{n=0}^{[(N-3)/2]}\sum_{p+q\le N+J-2n-3}[B_{pq}^{1,N-2n-3,r}\begin{pmatrix}V_{2n+3}^{pq11}(y)\\0\end{pmatrix}+B_{pq}^{2,N-2n-3,r}\times$$

$$\times\begin{pmatrix}0\\V_{2n+3}^{pq22}(y)\end{pmatrix}]-\sum_{n=0}^{[(N-5)/2]}\sum_{p+q\le N+J-2n-5}[B_{pq}^{1,N-2n-5,r}\begin{pmatrix}0\\V_{2n+5}^{pq12}(y)\end{pmatrix}+$$

$$+B_{pq}^{2,N-2n-5,r}\begin{pmatrix}V_{2n+5}^{pq12}(y)\\0\end{pmatrix}]$$

It is assumed that in (6), (7) only the sums where the upper limits are not less than the lower limits are considered.

The equations (6), (7) are solved sequentially for $N = 0, 1, 2$ etc. For every N equations (6), (7) correspond to the solution of static problem with polynomial loads. The equations (6), (7) are equalities between polynomials. Equating the coefficients of these polynomials one obtains a sequence of linear equation systems relatively unknown constants B_{pq}^{jmr}, B_{pq}^{jmi}. The developed method in case $\beta = 0$ leads to the constructive procedure for the analytical solution of static problem for elliptical crack under arbitrary polynomial loads. The detailed description of the proposed analytical procedure in case of static polynomial loads is presented in Kaptsov and Shifrin (1), (2). In case $\beta \neq 0$ the method leads to the analytical calculation of arbitrary amount of Taylor's series expansion coefficients of the solutions of (1), (2). Thus the developed procedure generalizes a number of papers where analytical solutions for static problems were obtained and only several coefficients of Taylor's series were culculated for some time-harmonic problems.

NUMERICAL RESULTS

To obtain the high accuracy approximation of the solutions of equations (1), (2) in the range of intermediate frequencies Pade approximants are used. The detailed description of Pade approximants and the conditions for their convergence is presented in Baker and Graves-Morris (3). Here we only remaind that there is exist an infinite matrix of Pade approximants [L/M]. For the convergence of Pade approximants a sequence of diagonal elements of the matrix [L/L] or parallel to them [L/L $\pm l$] have to be taken. For the special case of uniform normal time-harmonic load the outlined method was developed in Kaptsov and Shifrin (4). Here consider the problem for penny-shaped crack of radius a under shear time-harmonic loads. Let the amplitudes of applied load be $t = (1, 0, 0)$. The stress intensity factors at the point $x\prime$ of the crack front for the dimensionless wave number βa denote via $K_{II}(\beta a, x\prime)$ and $K_{III}(\beta a, x\prime)$. The graphs of normalized stress intensity factors $K^*_{II}(\beta a, x\prime) = |K_{II}(\beta a, x\prime)|/K_{II}(0, x\prime)$ at the point $x\prime = (a, 0, 0)$ and $K^*_{III}(\beta a, x) = |K_{III}(\beta a, x)|/K_{III}(0, x)$ at the point $x = (0, a, 0)$ for Poisson's ratios $\nu = 0.1$, $\nu = 0.3$ and $\nu = 0.5$ are depicted in Figs 1 and 2. The problem concerning stabilization of numerical results with increase of Pade approximants order will be discussed in the next publications.

REFERENCES

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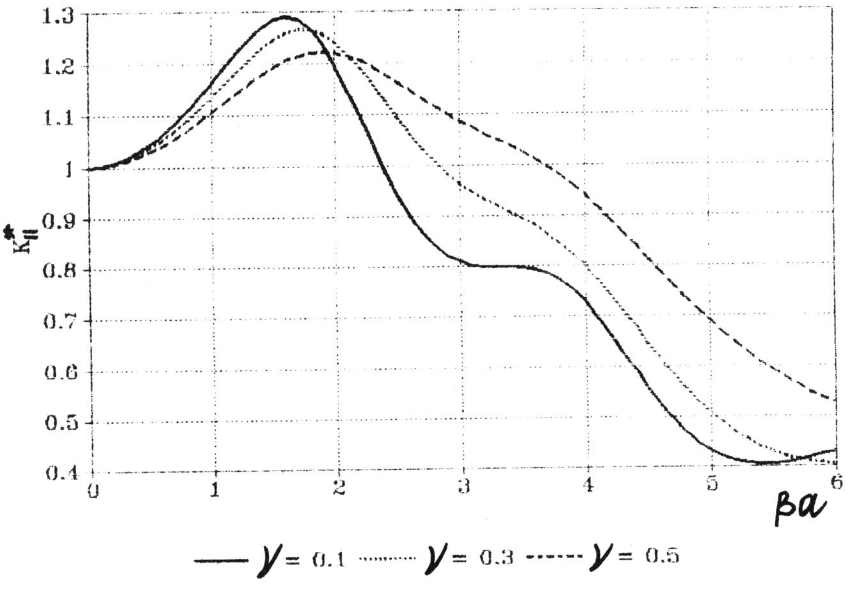

Figure 1 Normalized K_{II}

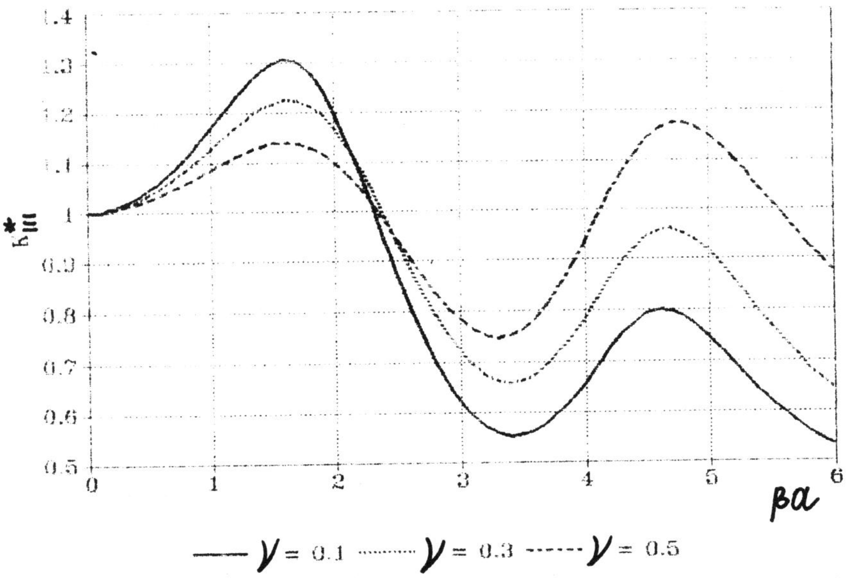

Figure 2 Normalized K_{III}