STRESS INTENSITY FACTORS IN STATICALLY INDETERMINATE CRACKED BEAMS

A. Valiente, M. Elices and F. Ustáriz*

This paper presents a method for calculating the stress intensity factors in cracked beams. The cracked cross-sections are treated as singular sections with finite bending and shear compliances from which the expressions of the stress intensity factors are derived. The principles of Strength of Materials are used for the structural analysis of statically determinate and indeterminate cracked beams by incorporating the effects of the cracks through such compliances. The method is applied for a particular case and the results are compared with that obtained with a finite element modelling.

INTRODUCTION

Cracking phenomena due to fatigue, welding or stress corrosion not only affect components used in advanced technologies, such as pipelines, pressure vessels or aircraft fuselages. Beams and other simple structural members also can become cracked and suffer brittle fracture with catastrophical consequences, as reported, for instance, by Rolfe and Barsom (1). However, the analysis of frames with cracked beams is an engineering problem not yet solved.

Fracture mechanics provides useful tools for the mechanical analysis of cracked bodies, but except for a few simple cases where the stress intensity factors can be obtained from handbooks, the analysis of a cracked beam must be performed in a numerical way, involving high costs and providing no general results. For example, in the method developed by M.H. El-Hadad et al (2) for the analysis of statically indeterminate plane frames containing cracks, the stiffness decreases of beams due to cracks are included in the stiffness matrix of the structure, but they have to be obtained from some available expression of the corresponding stress intensity factor.

* Department of Materials Science. Polytechnical University of Madrid

This work presents a simplified approach to calculate the stress intensity factor of statically indeterminate cracked beams in modes I and II. The bases of the method are the bending and shear compliances of a cracked cross-section. These compliances are numerically evaluated from the flexural and shear stresses that the theory of Strength of Materials provides for the section, incorporated into the classical Mohr's theorems for determining the unknown forces or moments, and used to derive expressions for the stress intensity factors.

THEORETICAL FORMULATION

In the analysis of beams based on Strength of Materials, local singularities such as concentrated forces, supports, hinges, etc., are reflected as local discontinuities in the shear and moment diagrams or in the deflection curve. This treatment can be extended to the case of cracked beams if the effect of cracks is assumed to be concentrated in the cracked sections, causing a discontinuity of slope and deflection in the elastic curve. The principle of superposition can be applied to isolate the effects of the crack and to evaluate them in terms of slope and deflection discontinuities. As Figure 1 indicates, the stresses and strains caused by the shear force and the bending moment near the cracked section result from combining those originated by three loading states: (0) the same force and moment acting in the absence of crack, (1) and (2) normal and shear stress distributions applied on the crack surfaces, which cancel out those along the crack plane in the state (0). Therefore, the applied stresses of the states (1) and (2) are the opposed to those caused by the bending moment and the shear force on the uncracked section and their values can be estimated from the theoretical solutions of Strength of Materials:

$$\sigma = \frac{M}{I} y \qquad (1) \qquad\qquad \tau = \frac{Q}{I} \frac{M(y)}{B(y)} \qquad (2)$$

where σ and τ are the normal and shear stress at a distance y from the neutral axis, M is the bending moment, Q is the shear force, I is the moment of inertia of the section, B(y) is the width of the section at the distance y, and $M(y)$ is the first moment about the neutral axis of the part of the section at a distance from the neutral axis greater than y.

The principle of superposition, as applied in Figure 1, shows that the effects of the crack on the beam are those caused by the loads of states (1) and (2). According to Figure 2, these effects can be evaluated as slope and deflection discontinuities in the elastic curve of the beam. The deformation of the beam under these loads is concentrated near the cracked section, so that beyond the affected region, the two portions of the beam at each side of the crack behave as rigid bodies. Because of symmetry of loading in state (1) and anti-symmetry in state (2), the motions of the two rigid parts of the beam in relation to each other are a rotation of angle Θ in the first case, and a translation of magnitude V normal to the neutral plane in the second. Neglecting the length of the deformed zone, these two motions are the only effects on the beam due to the crack, and represent discontinuities of slope and deflection at the cracked section for the elastic curve. On the other hand, according to Equations (1) and (2), the loads that cause the discontinuities, and as a consequence the discontinuities themselves are respectively proportional to the bending moment and to the shear force at the cracked section. Therefore, if θ is the slope of the elastic curve and v is the deflection:

$$\Delta\theta \equiv \Theta = C_m M \quad (3) \qquad\qquad \Delta v \equiv V = C_q Q \quad (4)$$

The proportionality factors C_m and C_q in Equations (3) and (4) are specific functions of the crack depth for each cross-section and can be considered as the bending and shear compliances of the cracked section. The dependence between these compliances and the crack depth a can be expressed in a dimensionless manner as:

$$C_m = \frac{W}{EI} m(a/W) \quad (5) \qquad\qquad C_q = \frac{W}{EA} q(a/W) \quad (6)$$

where W and A are respectively the width and the area of the entire cross-section, E is the Young´s modulus, and $m(\cdot)$ and $q(\cdot)$ are dimensionless functions only dependent on the cross-section configuration.

Under the loadings of the states (1) and (2), the crack will be deformed in modes I and II, respectively, because the loads of these states are symmetrically and anti-symmetrically distributed about the crack plane. Therefore, the stress intensity factors K_I and K_{II} can be calculated from the bending moment M and the shear force Q at the cracked section, and from the compliances C_m and C_q, through the well-known formulae:

$$K_I = \sqrt{G_I E} = M\sqrt{\frac{E}{2}\frac{dC_m}{dA_f}} = M\sqrt{\frac{E}{2B(a)}\frac{dC_m}{da}} = M\sqrt{\frac{m'(a/W)}{2IB(a)}} \quad (7)$$

$$K_{II} = \sqrt{G_{II} E} = Q\sqrt{\frac{E}{2}\frac{dC_q}{dA_f}} = Q\sqrt{\frac{E}{2B(a)}\frac{dC_q}{da}} = Q\sqrt{\frac{q'(a/W)}{2AB(a)}} \quad (8)$$

where A_f is the cracked area of the cross section, and $m'(\cdot)$ and $q'(.)$ are the derivatives of the functions $m(\cdot)$ and $q(\cdot)$.

To calculate the stress intensity factors it is necessary to determine the values of the bending moment M and the shear force Q. If the beam is statically indeterminate they would have to be found from compatibility of displacements and rotations, including those caused by the crack. These contributions to displacements and rotations can be added directly in the Mohr´s theorems as local increments of the slope and the deflection of the elastic curve at the cracked section. Hence, the problem can be solved within the context of Strength of Materials, with no additions other than the bending and shear compliances of the cracked cross-section. The difference of deflections $v_B - v_A$ and slopes $\theta_B - \theta_A$ between the ends A and B of a portion of the cracked beam containing the crack will be given by the expressions:

$$\theta_B - \theta_A = MC_m + \frac{1}{EI}\int_A^B M_f dx \tag{9}$$

$$v_B - v_A = \theta_A(x_A - x_B) + QC_q + MC_m(x_C - x_B) + \frac{1}{EI}\int_A^B (x - x_B)M_f dx + \frac{M_B - M_A}{GA_1} \tag{10}$$

The rotation between A and B is given by the two terms in Equation (9), which are the rotation at the cracked section and that due to the bending moments acting on the uncracked sections, respectively. The second and the fourth terms in Equation (10) are the analogous contributions for the deflection between A and B, the first and the third one are the deflections due to the rotations at the end A and the cracked section, respectively, and the last one is that produced by the shear force acting on the uncracked sections; G is the shear modulus, and A_1 is the shear effective area of the cross-section. Figure 3 illustrates both Equations.

ANALYSIS OF A STATICALLY INDETERMINATE CRACKED BEAM

The stress intensity factors of a cracked beam with rectangular cross-section such as the one depicted in Figure 4 were calculated using the formulated approach and directly, with a commercial finite element program. The span and the crack depth were respectively 10 and 0.4 times the width. The beam was modelled by means of a 2D mesh. The stress intensity factors were obtained from the displacements of the crack lips near the tip, but this procedure did not allow a reliable K_{II}-value to be obtained because the parallel displacements to the crack lips were excessively deviated from the theoretical field.

The functions m(·) and q(·) of a rectangular cross-section were numerically determined by solving the loading cases of the states (1) and (2) in Figure 2 for 9 crack depths ranging from 0.1 to 0.9 times the width of the section. These calculations were performed with the same finite element program that for the analysis of the cracked beam. The resulting values of m(·) were compared with that given by Tada et al (3), the maximum difference being lower than 8%. The 9 calculated values of the functions m(·) and q(·) were used to fit analytical expressions for each function.

Equation (10) allows the reaction force at the support to be found, and from this force the bending moment and the shear force at the cracked section can be obtained. Finally, the stress intensity factors were calculated by introducing the analytical expressions of m(·) and q(·) into Equations (7) and (8). Some significant values obtained by both methods are given in Table 1 for comparison.

TABLE 1– Values obtained in the analysis of the cracked beam of Figure 4. (B is the beam thickness)

	Present Approach	Finite Elements
Bending Moment (at the cracked section)	$0.42pW^2$	$0.43pW^2$
Shear Force (at the cracked section)	$0.057pW$	$0.058pW$
K_I	$3.63\frac{p}{B}W^{1/2}$	$3.67\frac{p}{B}W^{1/2}$
K_{II}	$0.073\frac{p}{B}W^{1/2}$	–

DISCUSSION AND CONCLUSIONS

Table 1 confirms the possibilities of the proposed approach. The differences between the values based on the approach and a refined finite element computation are lower than 3%, the same magnitude of error that can be expected in analogous analysis of the uncracked beam. The main advantages of the method are two: 1) the crack analysis has only to be performed for each cross-section configuration and not for each beam; 2) once the functions m(·) and q(·) of the given cross-section are known, the analysis can be applied to statically determinate or indeterminate cases using the procedures of Strength of Materials.

REFERENCES

(1) Rolfe, S.T. and Barsom, J.M., "Fracture and Fatigue Control in Structures", Prentice-Hall, Inc., Englewood Cliffs, New Jersey, U.S.A., 1977.

(2) El-Haddad, M.H., Ramadan, O.M. and Bazaraa, A.R., Int. J. Frac., Vol 45, 1990, pp. 81-102.

(3) Tada, H., Paris, P. and Irwin, G., "The Stress Analysis of Cracks Handbook" Del Research Corporation, St. Louis, Missouri, U.S.A., 1985

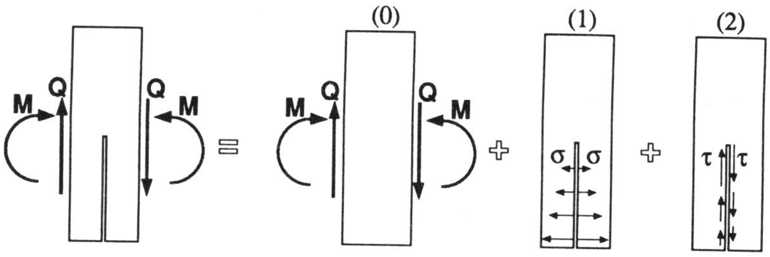

Figure 1. Principle of superposition applied to a cracked beam.

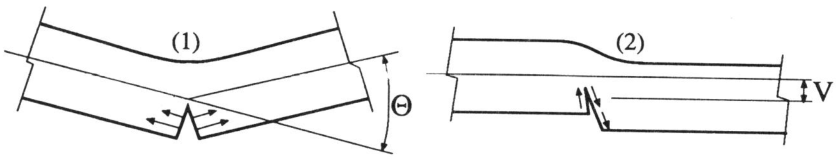

Figure 2. Beam deformations due to a crack.

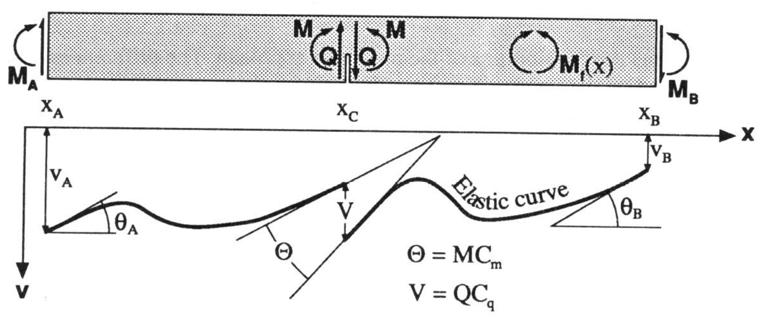

Figure 3. Structural analysis of cracked beams.

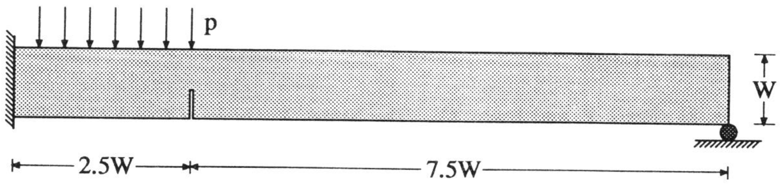

Figure 4. Statically indeterminate cracked beam used for applying the method.