Crack Propagation in a Composite Laminated Plate under Bending

Viktor Bozhydarnyk, Vasyl Shvabyuk, Iaroslav Pasternak, Volodymyr Shvabyuk

Lutsk National Technical University, Lvivska str. 75, 43018 Lutsk, Ukraine,
e-mail: shvabyuk@lutsk-ntu.com.ua

ABSTRACT. *The study of thin plates weakened by cracks is especially important in the case of composite materials, due to the possibility of interlayer delaminating. Crack growth parallel to the median surface of a plate under bending is less dangerous than the perpendicular crack propagation; however, the analysis of such defect's evolution is of great interest and has its possible applications in engineering analysis of fracture and fatigue of composite plates. In the present study, the bending of a circular plate containing a penny-shaped internal crack is considered based on the equations of the improved theory of the middle thickness plate bending. The influence of a transverse anisotropy and a length of the crack on a stress and displacement of the plate are analyzed.*

INTRODUCTION

This paper considers bending of a circular transversely isotropic plate, containing an internal penny-shaped crack, which is parallel to the median surface. Similar problems of bending, stability and vibration of cracked Kirchhoff-Love plates were considered earlier by Yeghiazaryan [1], Marchuk and Khomyak [3], Serensen and Zaytsev [4], Cherepanov [5] and others [7]. However, abovementioned solutions do not consider anisotropy and transverse compression of the plate. The stress intensity factors are also neglected due to used one-dimensional models, and hence, these solutions cannot be applied to analysis of fracture initiation and propagation.

Therefore, this paper utilizes the improved theory of bending [6], which accounts transverse shear and compression. This allows to account transverse anisotropy of the plate and to study stress intensity induced by the crack.

FORMULATION OF THE PROBLEM

Consider a circular plate of a radius R and a thickness $2h$ under a surface pressure q distributed uniformly at the face $z=-h$ (Fig. 1). At the distance $h_0 \in [0;h]$ from the bottom face of the plate, the latter contains a penny-shaped crack of a radius $l \in [0, R]$. The crack is parallel to a median surface of the plate.

To solve the stated problem one can utilize the technique [1], according to which the plate is formally decomposed into two domains with different bending rigidities:

- a domain containing the crack, which cylindrical rigidity equals the algebraic sum of rigidities of the upper and lower plate elements:

$$D_1 = D_1^- + D_1^+ = \delta D \text{, } (\delta = 1 - 3\beta + 3\beta^2,\ \beta = h_0/2h). \tag{1}$$

  (Here $D_1^+ = \tilde{E}(2h - h_0)^3/12 = (1-\beta)^3 D$ is a rigidity of the upper plate part above the crack; and $D_1^- = \beta^3 D$ is a rigidity of the lower plate part below the crack; h_0 is a distance from the bottom face of the plate to the crack; $\tilde{E} = E/(1-\nu^2)$; E is an elasticity modulus and ν is a Poisson ratio);
- and a domain without a crack, which cylindrical rigidity equals the rigidity of the unnotched plate $D_2 \equiv D = 2\tilde{E}h^3/3$.

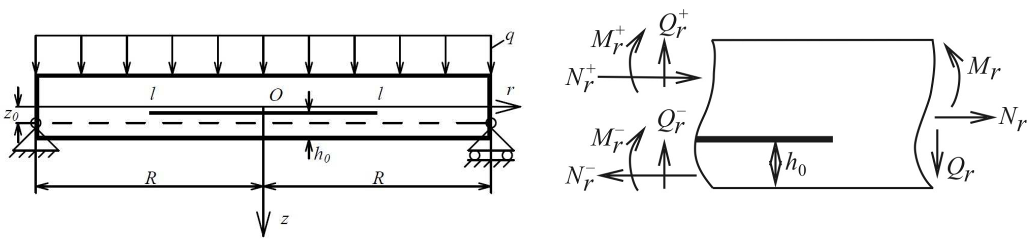

Figure 1. The sketch of the considered problem

It should be noted that the technique [1] can be applied in cases, when the plate model does not take into account the transverse compression, i.e. when vertical displacements do not depend on the transverse coordinate z. Within this technique it is impossible to determine the real radial stress σ_r, which act in the upper and lower parts of the plate over and under the crack, respectively. Therefore, henceforward the model of plates of a middle thickness [6], which utilize the improved equations of bending, is used.

SOLUTION STRATEGY

The differential equations of bending of transversely isotropic plates under uniformly distributed load can be written in the cylindrical coordinate system as follows [6]:

$$D_i \Delta^2 w_i = q_{i2} - \varepsilon_1 h_i^2 \Delta^2 q_{i2} - \varepsilon_2 h_i^4 \Delta^2 q_{i2}, \tag{2}$$

$$K_i' \Delta w_\tau^{(i)} = -q_{i2}\text{; } \quad \Delta u_i - \frac{u_i}{r^2} = -\frac{A'}{\tilde{E}} \frac{dq_{i1}}{dr},$$

where $D_i = \begin{cases} D_1 & r \in (0;l) \\ D_2 & r \in (l;R) \end{cases}$, $i=1$ for $r \in [0;l)$, and $i=2$ for $r \in (l;R]$; $\Delta \equiv \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}$;
$K'_i = \frac{4}{3}G'h_i$; $\varepsilon_\tau = \frac{2}{5}\frac{\tilde{E}}{G'}$; $D_i = D_2 = I\tilde{E}$; $I = 2h^3/3$; $K'_i = K' = 4G'h/3$; $q_{i1} = -0,5q^-$;
$w_\tau^{(i)} = -\frac{5}{4}\varepsilon_\tau h_i^2 \Delta w_i - \frac{\varepsilon_1}{K'_i} h_i^2 q_{i2} - \frac{\varepsilon_2}{K'} h_i^4 \Delta q_{i2}$; $q_{i2} = q^-$; $u_i, w_i, w_{i\tau}, h_i = u, w, w_\tau, h$ for the domain $r > l$; $h_i = h^+ = h(1-\beta)$; $\beta = h_0/2h$; $q_{i1} = q_{u1} = 0,5(\sigma_z(h-h_0) - q^-)$, $q_{i2} = q_{u2} = q^- + \sigma_z(h-h_0)$, $u_i, w_i, w_{i\tau} = u^+, w^+, w_\tau^+$; $D_i = D_1^+ = I^+\tilde{E}$; $I^+ = 2h^3(1-\beta)^3/3$, $K'_i = K'_u = 4G'h(1-\beta)/3$ or $D_i = D_1^- = I^-\tilde{E}$; $I^- = h_0^{\ 3}/12$; $K'_l = 2G'h_0/3$; $q_{i1} = q_{l1} = 0,5\sigma_z(h-h_0)$; $q_{i2} = q_{l2} = -\sigma_z(h-h_0)$ and $u_i, w_i, w_{i\tau}, h_i = u^-, w^-, w_\tau^-, h_0/2$ for the top and bottom parts of the plate in the domain $r \le l$, respectively; $\varepsilon_1 = \frac{2}{5}(1-0,75\nu^*)\frac{\tilde{E}}{G'}$; $\varepsilon_2 = \frac{1}{20}(1-\nu^*)\frac{\tilde{E}}{E'}$; $\tilde{E} = E/(1-\nu^2)$; $A' = \frac{\nu''}{1-\nu}$; $\nu^* = 0,5\nu''G'/G$; E, E', G, G', ν, ν'' are the elastic and shear moduli and Poisson ratios of the plate in the longitudinal and transverse (with primes) directions; $q^- = q = const$ is the distributed load applied to the top surface of the plate ($z=-h$); u_i are horizontal displacements of the median surfaces of the upper and lower parts of the plate; w and w_τ are the entire and shear vertical displacements of the median surface of an uncracked part of the plate; the Roman numerals at superscripts of w, w_τ, u and q_1, q_2 denote the order of a derivative on the variable r; subscripts «*u*» and «*l*» denote respectively upper and lower parts of the plate at the cracked domain; $2h$ is a thickness of the plate; h_0 is a thickness of the plate part which is under the crack. In the formulated problem one assumes that the bottom face ($z=h$) of the plate is traction free, hence, $q^+ \equiv 0$, and the stress $\sigma_z(h-h_0)$ equals to the contact pressure between crack faces.

One can obtain the value of normal contact pressure p (or the stress $\sigma_z(h-h_0)$) within the framework of the Kirchhoff – Love hypotheses for thin plates, or based on the Timoshenko plates theory. Both states that vertical displacement w (together with their derivatives) does not depend on the transverse coordinate z, i.e. $w_l = w_u = w$. Therefore, the first equation of the system (2) for the upper and lower parts of the plate can be written as:

$$D_1^+ \Delta^2 w = q_{u2} = q - p; \quad D_1^- \Delta^2 w = q_{l2} = p. \tag{3}$$

Thus, the approximate value of the contact pressure between crack faces, according to Eq. (3), is equal to

$$(D_1^+ + D_1^-)\Delta^2 w = q; \quad p = \frac{qD_1^-}{D_1^+ + D_1^-} = q\beta^3/\delta, \tag{4}$$

where $\delta = 1 - 3\beta + 3\beta^2$.

Hence, the normal contact stress acting on crack faces equals

$$\sigma_z(h - h_0) = -p = -q\beta^3 / \delta . \quad (5)$$

Stresses σ_r and σ_z, and displacements $U(r,z)$ and $W(r,z)$ of a plate in the uncracked domain ($r \in (l;R]$) , according to the model [6], are as follows

$$\sigma_r = \frac{N_r}{2h} + \frac{M_r}{I} z + \frac{zG^*}{3I}\left(z^2 - 0.6h^2\right)\left(q_2 - \Lambda q_2 h^2 \frac{G'}{E'}\right); \quad (6)$$

$$\sigma_z = q_1 + \frac{1}{4}\left(3\frac{z}{h} - \frac{z^3}{h^3}\right)\cdot q_2;\ q_1 = \frac{1}{2}(q^+ - q^-),\ q_2 = (q^+ + q^-);$$

$$U(r,z) = u(r) - z\left(\frac{dw}{dr} - \frac{dw_\tau}{dr}\left(1 - \left(1 - \nu^*\right)\frac{z^2}{3h^2}\right)\right) - \frac{\left(1 - \nu^*\right)}{8E'h}\frac{dq_2}{dr} z^3 ;$$

$$W(r,z) = \omega(r) + 2\alpha_0 z \cdot \frac{q_1}{E'} + A' \cdot \frac{z^2}{2}\Delta w + \frac{\alpha_0 \cdot q_2}{8E'h}\cdot B(z), \quad (7)$$

where $A' = \dfrac{\nu''}{(1-\nu)}$; $B(z) = 6B_2 z^2 - B_3\dfrac{z^4}{h^2}$; $G^* = \dfrac{1}{4(1-\nu)}\left(\dfrac{E}{G'} - \nu''(3+\nu)\right)$;

$\varepsilon_0 = \dfrac{1}{20(1-\nu)}\left(4\dfrac{E}{G'} - \nu''(7-\nu)\right)$; $\tilde{w} = w + 1.5\varepsilon_2 q_2 h / \tilde{E}$, $B_2 = 1 + \dfrac{A'E'}{2\alpha_0 G'}$;

$B_3 = B_2 - \dfrac{\nu'' A' E'}{4\alpha_0 G}$; $\alpha_0 = 0.5 - \nu' \cdot A'$; $M_r = \int\limits_{-h}^{h} z\sigma_r dz = -D(\dfrac{d^2 w}{dr^2} + \dfrac{\nu}{r}\dfrac{dw}{dr}) - \varepsilon_0 h^2 q_2$,

$N_r = \int\limits_{-h}^{h} \sigma_r dz = 2\tilde{E}h(\dfrac{du}{dr} + \nu\dfrac{u}{r}) + 2A'hq_1$, $Q_r = K'\dfrac{dw_\tau}{dr}$ are the bending moment, normal and shear forces, respectively; $\Lambda = \dfrac{1}{1+\nu}\left(\dfrac{d^2}{dr^2} + \dfrac{\nu}{r}\dfrac{d}{dr}\right)$ is a differential operator; u is a tangential displacement of the median surface of the uncracked domain of the plate.

The general solution of Eqs. (2) and (4) has the following form

$$w_i = A_i r^2 + \tilde{B}_i r^2 \ln\frac{r}{R} + C_i \ln\frac{r}{R} + K_i + \frac{q_{i2} r^4}{64 D_i},\ (i = 1,2)$$
$$w_\tau^{(i)} = A_\tau^{(i)} + B_\tau^{(i)} \ln\frac{r}{R} - q_{i2} r^2 / 4K_i',\ u_i = F_i r + L_i r^{-1} + u_i^*, \quad (8)$$

where integration constants $A_i, \tilde{B}_i, C_i, K_i, A_\tau^{(i)}, B_\tau^{(i)}, F_i, L_i$ are determined from the boundary conditions, and u_i^* are the particular solutions.

From the displacement and bending moment boundedness conditions it follows, that four integration constants are equal to zero: $\tilde{B}_1 = C_i = B_\tau^{(1)} = L_i = 0$. The notations used here should be extended to account values corresponding to the upper and lower parts of the plate. Thus, these faces will be denoted with signs «+» and «-», respectively. Then for the cracked domain $(0,l)$ of the plate displacements (8) for the plate part under the crack can be written as,

$$w_1^- = A_1^- r^2 + K_1^- + \frac{pr^4}{64D_1^-}, \quad w_\tau^- = A_\tau^- - pr^2/(4K_l'), \quad u_1^- = F_1^- r + \frac{\nu'' p}{2E} r . \tag{9}$$

For the uncracked domain $(l;R)$ displacements are equal to:

$$w_2 = A_2 r^2 + \tilde{B}_2 r^2 \ln\frac{r}{R} + K_2 + \frac{qr^4}{64D_2}; \tag{10}$$

$$w_\tau^{(2)} = A_\tau^{(2)} + B_\tau^{(2)} \ln\frac{r}{R} - qr^2/4K', \quad u_2 = F_2 r + \frac{\nu'' q}{2E} r .$$

Integration constants $A_i, \tilde{B}_2, K_i, \ B_\tau^{(2)}, A_\tau^{(i)}, F_i$ are determined within the boundary conditions at the edge $r = R$ of the plate. In the case of a hinge supported plate these boundary conditions write as,

$$w_2(R) = 0, \quad w_\tau(R) = 0, \ M_{r2}(R) = 0, \ N_{r2}(R) = 0 . \tag{11}$$

Satisfying conditions (11) one can obtain that

$$A_2 = -\frac{qR^2(3+\nu)}{32D(1+\nu)} - \frac{\varepsilon_0 h^2 q}{2D(1+\nu)}; \ A_\tau^{(2)} = q_2 R^2/4K'; \ F_2 = 0;$$

$$K_2 = \frac{qR^4}{64D}\frac{5+\nu}{1+\nu}\left(1 + \frac{32\varepsilon_0}{5+\nu}\frac{h^2}{R^2}\right), \ M_{r2} = \frac{qR^2}{16}(3+\nu)(1-\frac{r^2}{R^2}); \tag{12}$$

$$Q_{r2}(R) = K'\frac{dw_\tau^{(2)}}{dr}\bigg|_{r=R} \equiv -D\frac{d}{dr}(\Delta w_2)\bigg|_{r=R} = -\frac{qR}{2} .$$

Here it is assumed that the constants $B_2, B_\tau^{(2)}$ can be determined from the equilibrium of the shear forces Q_r and under the given load they are zero ($\tilde{B}_2 = B_\tau^{(2)} = 0$).

Except the conditions (11) at the edge of the plate, it is necessary to satisfy the contact conditions between the cracked and uncracked domains at $r = l$:

$$w_1^-(l) = W(l, h(1-\beta)); \ \sigma_r^-(l, \beta h) = \sigma_r(l, h); \quad N_r^-(l) = \int_{h-h_0}^{h} \sigma_r(l,z)dz , \tag{13}$$

where $$\sigma_r^-(r, z_l) = \frac{N_r^-(r)}{h_0} + \frac{M_r^-(r)}{I^-} z_l + \frac{z_l G^*}{3I^-}\left(z_l^2 - 0.15h_0^2\right)\left(q_{l2} - 0.25\Lambda q_{l2} h_0^2 \frac{G'}{E'}\right);$$

$z_l = z - h + h_0/2$ is a local coordinate of the plate part under the crack, which is directed downwards.

Hence, the normal and shear forces and the bending moment acting at the bottom part of the plate under the crack are defined as [6],

$$N_{r1}^- = h_0 \tilde{E}[(u_1^-)' + \nu u_1^-/r] + h_0 A' q_{l1} ,$$

$$Q_{r1}^- = K_l' \frac{dw_\tau^-}{dr}, \ M_{r1}^- = -D_1^-\left(\frac{d^2 w^-}{dr^2} + \frac{\nu}{r}\frac{dw^-}{dr}\right) - 0.25\varepsilon_0 q_{l2} h_0^2 .$$

Here $M_{r1}^{-}(r)=M_{r1}^{-}(l)+\dfrac{3+\nu}{16}\left(l^2-r^2\right)p$; $Q_{r1}^{-}(l)=Q_{r2}(l)+q_{u2}l/2$; $Q_{r2}(l)=-ql/2$.

Satisfying the boundary conditions (13) at the interface $r=l$ one can obtain equations for determination of the rest of unknown integration constants:

$$A_1^{-}l^2-tA_2l^2+K_1^{-}-K_2+\frac{ql^4}{64D\delta}(1-\delta-\delta')+\frac{qh}{8E'}B(\beta)=0,$$
$$N_{r1}^{-}(l)h/\beta+3M_{r1}^{-}(l)/\beta^2=3M_{r2}(l)+0.4(q-p)h^2G^*; \qquad (14)$$
$$N_{r1}^{-}(l)=3\beta(1-\beta)M_{r2}(l)/h-0.1qhG^*f/4,$$

where $F_1^{-}=\dfrac{3M_{r2}(l)}{2Eh^2}(1-\nu)(1-\beta)-\dfrac{0.1qG^*f}{8E\beta}(1-\nu)$; $G^*=\dfrac{1}{4(1-\nu)}\left(\dfrac{E}{G'}-\nu''(3+\nu)\right)$;

$B(\beta)=\alpha_0(1-\beta)[8-f_1(\beta)B_2-\tilde{A}_2]$; $\delta'=8\delta A'(1-\beta)^2h^2/l^2)=4\delta(t-1)$;

$t=1+2A'(1-\beta)^2h^2/l^2$; $\tilde{A}_2=\dfrac{(\nu'')^2(1-\beta)^3E'}{2(1-\nu-2\nu'\nu'')G}$; $M_{r2}(l)=\dfrac{qR^2}{16}(3+\nu)(1-\theta^2)$; $\theta=\dfrac{l}{R}$;

$f_1(\beta)=(1-\beta)(5+2\beta-\beta^2)$; $f=1-(1-2\beta)^2\left(1+20\beta(1-\beta)\right)$.

Solving the systems of equations (12) and (14) simultaneously, one can derive the integration constant A_1, the biggest deflection $w_1^{-}(0)$ of the bottom part of the plate at its center, and the moment $M_{r1}^{-}(l)$:

$$A_1=-\frac{qR^2}{32(1+\nu)D}\left[(3+\nu)\left(1+\theta^2(\frac{1}{\delta}-1)\right)+16\varepsilon_0\beta^2\frac{h^2}{\delta R^2}\right]-\frac{0,2(q-p)h^2G^*}{3(1+\nu)\beta D}-\frac{0,1qh^2G^*f}{24(1+\nu)\beta^2D};$$

$$w_1^{-}(0)=\frac{qR^4}{64D}\frac{5+\nu}{1+\nu}\left[1+\left(\frac{1}{\delta}-1\right)\theta^4-\frac{\delta'\theta^2(3+\nu)}{2\delta(5+\nu)}\left(1-2\theta^2\frac{1+\nu}{3+\nu}\right)+\frac{32\varepsilon_0}{5+\nu}\frac{h^2}{R^2}\left(1+\theta^2(\frac{\beta^2}{\delta}-t)\right)\right]+$$
$$+\frac{0,2\theta^2(q-p)R^2h^2G^*}{3(1+\nu)\beta D}+\frac{0,1\theta^2qR^2h^2G^*f}{24(1+\nu)\beta^2D}-\frac{qh}{8E'}B(\beta); \qquad (15)$$
$$M_{r1}^{-}(l)=\beta^3M_{r2}(l)+0,4\beta^2(q-p)h^2G^*/3+0,1\beta qh^2fG^*/12=$$
$$=\frac{q\beta^3R^2}{16}(3+\nu)(1-\theta^2)+0,4\beta^2(q-p)h^2G^*/3+0,1\beta qh^2fG^*/12.$$

Maximal stress $\sigma_{r1}^{-}(0,\pm h_0/2)$ can be obtained from Eqs. (6), (13) as

$$\sigma_{r1}^{-}(0,\pm h_0/2)=\frac{N_{r1}^{-}(l)}{2\beta h}\pm\frac{3M_{r1}^{-}(0)}{2\beta^2h^2}\pm 0,2G^*p, \qquad (16)$$

where $N_{r1}^{-}(l)=3\beta(1-\beta)M_{r2}(l)/h-0.1qhG^{*}f/4$; $M_{r2}(l)=\frac{qR^2}{16}(3+\nu)(1-\theta^2)$;

$M_{r1}^{-}(0)=\frac{3+\nu}{16}pl^2+\beta^3M_{r2}(l)+0.4\beta^2(q-p)h^2G^{*}/3+0.1\beta qh^2fG^{*}/12$;

$f(\beta)=1-(1-2\beta)^2(1+20\beta(1-\beta))$; $G^{*}=\frac{1}{4(1-\nu)}\left(\frac{E}{G'}-\nu''(3+\nu)\right)$.

Substituting the values of $N_{r1}^{-}(l)$ and $M_{r1}^{-}(0)$ into Eq. (15) one can obtain the closed-form formulae for the maximal stress σ_{r1}^{-} at the external surface of the plate under the crack:

$$\sigma_{r1}^{-}(0,h_0/2)=\frac{3(3+\nu)qR^2}{32h^2}\left[1+\left(\frac{\beta}{\delta}-1\right)\theta^2\right]+0.2G^{*}q; \quad (17)$$

$$\sigma_{r1}^{-}(0,-h_0/2)=\frac{3(3+\nu)qR^2}{32h^2}\left[(1-\theta^2)(1-2\beta)-\frac{\beta}{\delta}\theta^2\right]-0.2G^{*}q-0.025G^{*}qf(\beta)/\beta.$$

For determination of stresses in the plate part above the crack, one can utilize Eq. (6) in the local coordinates (z_u, r):

$$\sigma_r^{+}(r,z_u)=\frac{N_r^{+}}{h_0}+\frac{M_r^{+}}{I^{+}}z_u+\frac{z_uG^{*}}{3I^{+}}\left(z_u^2-0.6h^2(1-\beta)^2\right)\left(q_{u2}-0.6\Lambda q_{u2}h^2(1-\beta)^2\frac{G'}{E'}\right), \quad (18)$$

where $z_u=z+\beta h$ is a transverse coordinate of the upper part of the plate above the crack, directed downwards to its median surface.

The values of N_r^{+}, M_r^{+} are obtained from the contact conditions on the interface of cracked and uncracked domains at $r=l$:

$$N_r^{+}(l)=\int_{-h}^{h(1-2\beta)}\sigma_r(l,z)dz; \quad \sigma_r^{+}(l,-(1-\beta)h)=\sigma_r(l,-h). \quad (19)$$

Satisfying conditions (19), one obtains

$$N_{r1}^{+}(l)=-3\beta(1-\beta)M_{r2}(l)/h+0.1qhG^{*}f/4; \quad (20)$$

$$3M_{r1}^{+}(l)=h(1-\beta)N_{r1}^{+}(l)+3(1-\beta)^2M_{r2}(l)+0.4(1-\beta)^2ph^2G^{*}.$$

And hence,

$$M_{r1}^{+}(l)=(1-\beta)^3M_{r2}(l)+0.4(1-\beta)^2ph^2G^{*}/3+0.1(1-\beta)qh^2fG^{*}/12. \quad (21)$$

Consequently, maximal stress $\sigma_{r1}^{+}(0,\pm h_u/2)$ can be obtained from Eq. (18) with the account of Eqs. (20), (21),

$$\sigma_{r1}^{+}(0,\pm h_u/2)=\frac{N_{r1}^{+}(l)}{2(1-\beta)h}\pm\frac{3M_{r1}^{+}(0)}{2(1-\beta)^2h^2}\pm0.2G^{*}(q-p), \quad (22)$$

where $h_u=2h^{+}=2h(1-\beta)$; $M_{r1}^{+}(0)=M_{r1}^{+}(l)+\frac{3+\nu}{16}(q-p)l^2$.

The problem is solved under the assumption that the applied load causes crack faces to be in a smooth contact, thus, the opening mode stress intensity factor (SIF) K_{I} is equal to zero. At the same time, in front of the penny-shaped crack a shear stress exists even for the smooth contact of crack's faces, which causes nonzero values of a sliding mode SIF K_{II}. It is a challenging problem to find the latter based on the proposed improved theory of plates of average thickness. However, one can obtain the qualitative estimation of SIF using the approximate formula

$$K_{\mathrm{II}}\left(l,\beta\right)=\tau_{rz}^{0}\left(l,\beta\right)\sqrt{2\pi l}\,, \qquad (23)$$

where $\tau_{rz}^{0}\left(l,\beta\right)=-3\beta\left(1-\sqrt{\beta}\right)\dfrac{Q_r\left(l\right)}{h}=\sqrt{4.5}q\sqrt{\pi l}\theta\beta\left(1-\sqrt{\beta}\right)R/h$ is a shear stress in front of the crack.

In particular, for a case when the crack is placed at the median surface of the plate $\left(\beta=0.5\right)$, SIF K_{II} equals

$$K_{\mathrm{II}}\left(l,0.5\right)=0.3q\sqrt{\pi l}\cdot\theta R/h\,. \qquad (24)$$

Together with the Paris-like crack growth law this allows to simulate the internal fatigue crack propagation in the composite laminates.

To verify the obtained results the dual boundary element method is utilized. The boundary integral equations are adopted for studying of internal closed cracks. Special numerical quadratures, polynomial transformations and shape functions are utilized for accurate determination of the stress intensity factor. Numerical results are in good agreement with the analytic calculations.

REFERENCES

1. Yeghiazaryan, T.H. (2006) *Mechanics. Proc. of National Academy of Sciences of Armenia* **59** (2), 57-61.
2. Lekhnitskii, S.G. (1959) *Proc. of Academy of Sciences of USSR* **2**, 57-61.
3. Marchuk, M.V., Homyak, M.M. (2003) *On a mixed finite element analysis calculation of composite laminated shells and plates*, NAS of Ukraine, L'viv. (in Ukrainian).
4. Serensen, S.V., Zaytsev, G.P. (1982) *Bearing capacity of thin-slab structures built of reinforced plastics with defects*, Naukova Dumka, Kyiv (in Russian).
5. Cherepanov, G.P. (1983) *Fracture mechanics of composite materials*, Nauka, Moscow (in Russian).
6. Shvabyuk, V.I. (1980) *Int. Appl. Mech.* **16**(4), 322-328.
7. Shvabyuk, V., Sulym, H., Pasternak, Ia. (2011) *Acta mechanica et automatica* **5**(4), 94-100.