Towards a probabilistic concept of the Kitagawa-Takahashi diagram

Alfonso Fernández-Canteli[1], Roberto Brighenti[2], Enrique Castillo[3]

[1] Dept. of Construction and Manufacturing Engineering, University of Oviedo, Campus de Viesques, 33203 Gijón, Spain afc@uniovi.es
[2] Dept. of Civil-Environmental Engineering and Architecture, University of Parma, Viale G.P. Usberti 181/A, 43100 Parma, Italy brigh@unipr.it
[3] Dept. of Applied Mathematics and Computational Sciences, University of Cantabria, Avda de los Castros s/n , 39005 Santander, Spain castie@unican.es

ABSTRACT. *The Kitagawa-Takahashi (K-T) diagram, implemented by the El Haddad equation, relates the conventional fatigue limit to the crack size, enabling a boundary for the cyclic stress range to be established, below which an infinite life of the structural component may be, theoretically, ensured for any crack size due to the non-propagation of micro- and macrocracks. In order to account for the inherent random character of the fatigue phenomenon in real materials and the need of extending the K-T applicability to any prefixed number of cycles, advanced probabilistic S-N models should be considered to define the fatigue limit. In this way, a new basis towards a probabilistic Kitagawa-Takahashi-El Haddad approach is provided in agreement with the asymptotic matching proposed by Ciavarella-Monno.*

INTRODUCTION AND MOTIVATION

The Kitagawa-Takahashi (KT) diagram [1] represents a boundary in terms of crack size and stress range for which infinite fatigue lifetime of structural or mechanical components can be safely ensured due to non-propagating micro- and macrocracks. Such fatigue life assessment can be related to both the classical fatigue limit concept, resulting from the experimental-based *S-N* approach, and the threshold stress intensity factor range, as defined by the crack propagation law.

Even after the transcendent improvement provided by the *intrinsic crack* concept of El Haddad (EH) [2], two issues need to be dealt with: a) the extension of the KT-EH diagram to a fatigue limit for finite number of cycles, which is not necessarily identified with the endurance limit for $N=\infty$, and b) a stochastic definition of the KT-EH diagram as a consequence of the variability of the basic fatigue functions being considered (*S-N* and crack growth rate curves). Both represent practical requirements related to structural integrity design.

In this work, a new approach to the problem is supplied by considering the probabilistic *S-N* developed by Castillo and Fernández-Canteli [3], which provides a sound basis in the definition of the KT-EH line, permitting also the model to be extended from infinite to any finite life, as already proposed by Ciavarella and Monno [4], which provide an interesting alternative to solve the first issue, whereas the limitations of the simplistic *S-N* field used, based on a truncated Basquin approach with absence of probabilistic considerations, evidence the need of further enhancement.

Thus, the boundary between fracture and non-fracture can be optionally referred to a finite number of cycles for a certain probability of failure, because in the engineering practice a high, but finite, number of cycles, rather than infinity, are usually assumed for fatigue life.

PROBABILISTIC CONCEPTS APPLIED TO THE K-T APPROACH

In this work, only variability of the *S-N* field is considered, while a deterministic concept of the crack growth rate curve is for the present assumed [5] as a first attempt of introducing probabilistic considerations in the definition of the KT-EH diagram.

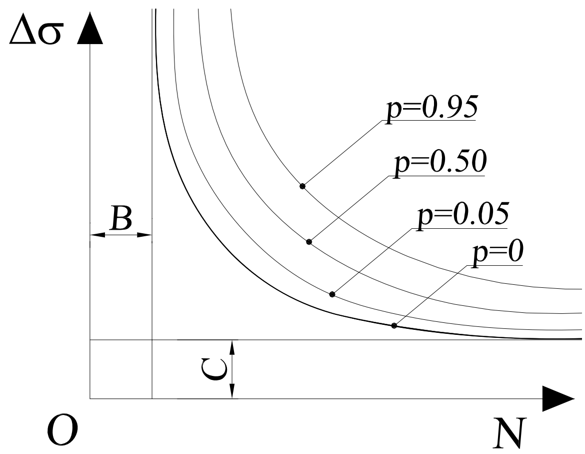


Figure 1. S-N field with percentile curves according to the regression model[3].

The proposed probabilistic model.

According to the probabilistic regression model proposed by Castillo and Fernández-Canteli [3] the fatigue life given stress range, and the stress range given lifetime are random variables following a Weibull distribution (Fig. 1):

$$p = 1 - \exp\left[-\frac{(\log N - B)(\log \Delta\sigma - C) - \lambda}{\delta} \right]^{\beta}, \qquad (1)$$

where p is the probability of failure, N the lifetime in cycles, $\Delta\sigma$ the applied stress range, B represents the threshold value of the lifetime and C the endurance limit, or fatigue limit for $N \to \infty$, and β, δ and λ are, respectively, the shape, scale and location Weibull parameters. As soon as the five parameters are estimated, the model provides a complete analytical description of the probabilistic *S-N* field being dealt with. The percentile curves are hyperbolas sharing the asymptotes $\log N = B$ and $\log \Delta\sigma = C$ (see Fig. 1), the zero percentile curve represents the minimum possible required number of cycles to achieve failure for different values of $\log\Delta\sigma$. The percentile curves can be interpreted as representing different initial flaw sizes.

Since the model extends the Wöhler field up to an unlimited number of cycles, an extrapolation of the lifetime is possible outside the range of number of cycles tested in the experimental program.

The main limitation of the model, consisting in the absence of an upper bound in the LCF region, may be overcome by considering a new fatigue variable, $\Delta\sigma\cdot\varepsilon_{\max}$, as the reference parameter controlling the fatigue process in the previous model. Once the model parameters in terms of $\Delta\sigma\cdot\varepsilon_{\max}$ have been estimated, the S-N field may be reconverted to the conventional $\Delta\sigma$ variable using the stress-strain cyclic diagram of the material, for instance as Ramberg-Osgood (R-O) curve.

K-T diagram for a finite number of cycles to failure N_L

Unless otherwise specified, the *S-N* curves are generally related to a unique probability of failure, *p*=0.5 or *p*=0.05, though the scatter of experimental data requires the definition of the whole *S-N* field as percentile curves based on statistical principles. The fatigue behaviour of a structural component is mainly governed by its surface (or volume) state quality, implying roughness and imperfections that are preferential sites for crack nucleation and subsequent propagation. According to [3], the percentile curves can be assumed to be associated with the probability of the existence of a crack being less than a certain initial crack size, a_i, initially unknown. Consequently, the fatigue failure is governed by the maximum crack size present in the specimen being tested so that percentile curves with increasing probabilities of failure are related to diminishing crack sizes: the percentile curve *p*=0, corresponding to the greatest, or worst, of the maximum crack sizes of the population, i.e., $a_{i,w}=\max(a_{\max})$, which is denoted *max-max crack size* (Fig. 2). Similarly, the upper percentile curve, *p*=1, corresponds to the minimum, or best, of the maximum crack sizes of the population, i.e., $a_{i,b}=\min(a_{\max})$, which is denoted *min-max crack size*. For practical purposes, the definition of the latter can be relaxed identifying $a_{i,b}$ as an initial crack size related to a high probability of failure, for instance, p_b=0.90 or 0.95. The two curves associated with $a_{i,w}$ and $a_{i,b}$ represent the two limiting sizes of the initial maximum defect corresponding to the particular surface finishing of the tested material. According to [3], $a_{i,w}$, related to the percentile curve *p=0*, is determined as

$$a_{i,w}=\frac{1}{\pi}\left(\frac{\Delta K_{th}}{Y\,\Delta\sigma_0(\infty)}\right)^2, \qquad (2)$$

where ΔK_{th} is the threshold SIF range, Y the geometric factor of the crack and $\Delta\sigma_0(\infty)$ the endurance limit.

For a given number of cycles to failure N_L, see Fig. 2, two different stress ranges $\Delta\sigma_b$ and $\Delta\sigma_w$ ($\Delta\sigma_b>\Delta\sigma_w$), are identified with the best and worst surface defects respectively. Consequently, for N_L as a reference lifetime, an *engineering SIF* *threshold* $\Delta K_{th,eng}$ can be found, particularly for the defect sizes $a_{i,w}$ and $a_{i,b}$, but also for any crack size $a_{i,m}$ ($a_{i,w} > a_{i,m} > a_{i,b}$), to be given by

$$\Delta K_{th,eng,m} = Y\sqrt{\pi\, a_{i,m}} \cdot \Delta\sigma_m \ , \tag{3}$$

where the stress range $\Delta\sigma_m$ results from the percentile curve corresponding to a_m (see Fig. 2). For $N_L \to \infty$, the $\Delta K_{th,eng}$ becomes the true threshold value ΔK_{th} of the crack growth rate curve.

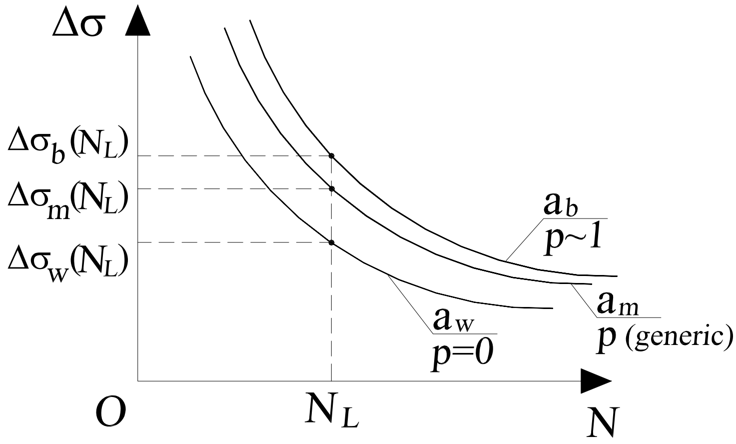


Figure 2. Probabilistic concept applied to an experimental S-N curve for a given material.

Note that, according to El Haddad [2], the intrinsic crack size is given as:

$$a_0 = \frac{1}{\pi}\left(\frac{\Delta K_{th}}{Y \cdot \Delta\sigma_0}\right)^2 , \tag{4}$$

where $\Delta\sigma_0$ represents in this case the fatigue limit for N_L number of cycles, pointing out that only for $N_L \to \infty$ the intrinsic initial crack size a_0 coincides with the worst crack size $a_{i,w}$. Applying the El Haddad equation to the evaluation of testing samples with varying surface states, i.e. intrinsic crack sizes a_0, would provide different fatigue limits $\Delta\sigma_0$, i.e., unlike KT diagrams, evidencing dependency with respect to the particular surface state tested, thus, proving that such KT diagrams are not a material characteristic. On the contrary, a unique KT diagram is expected from the proposed approach despite the different endurance limits, associated with the worst crack sizes a_w, obtained for the respective surface states.

The assumption of a deterministic crack growth rate curve implies a deterministic relation between crack size and number of cycles to failure meaning that the KT-EH diagram is deterministic too [5]. Accordingly, the scatter of the experimental data are only related to the initial surface state of the material, implying simply uncertainties related to measurement precision or to material heterogeneity.

For a certain initial crack size $a_{i,m}$, the failure condition, $K_{fm}=K_{Ic}$, resulting for a given number of cycles to failure, N_L, is equivalent to

$\Delta K(N) = Y \cdot \Delta\sigma_m \sqrt{\pi \cdot a_{fm}(N)} = \Delta K_{Ic}$ allowing us to express the final crack size ratio, $a_{f,w} / a_{f,m}$ as:

$$\frac{a_{f,w}}{a_{f,m}} = \left(\frac{\Delta\sigma_m}{\Delta\sigma_w} \right)^2 > 1. \tag{5}$$

Assuming a constant geometry factor *Y* during the crack propagation, the current crack size ratio r

$$r = \frac{a_w(N)}{a_m(N)} = \left(\frac{\Delta\sigma_m}{\Delta\sigma_w} \right)^2 \tag{6}$$

remains constant along the whole propagation process (see Fig. 3(b)), in particular, from the beginning of the fatigue process, i.e., for the initial crack sizes $a_m = a_{i,m}$ and $a_w = a_{i,w}$, up to the final failure state, proving that on a logarithmic scale, the vertical distance $c = \log r$ between the curves related to $a_{i,w}$ and $a_{i,m}$ remains constant all along the propagation process (see Fig. 3(b)).

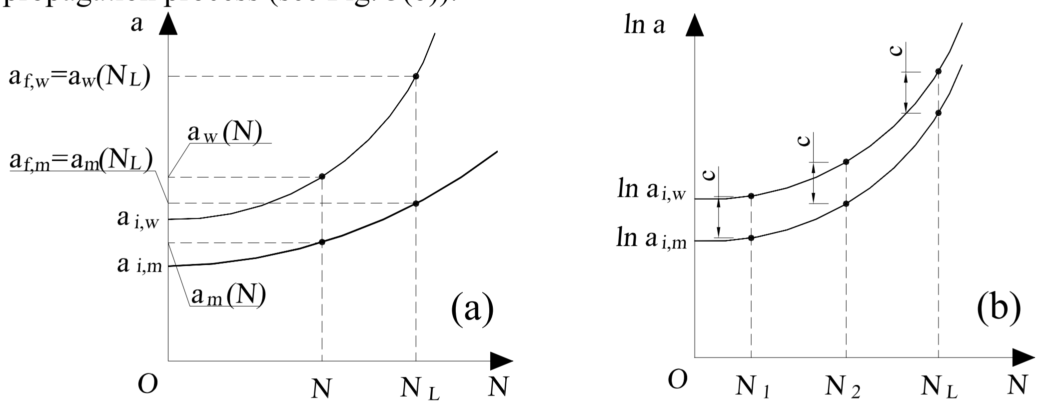


Figure 3. Crack growth curves for the best and worst initial defect cases for a given material (a) plotted in natural scale, and (b) plotted in logarithmic scale.

While the initial worst crack size $a_{i,w}$, related to *p*=0, is determined from Eq. (2), another crack size, $a_{i,m}$, related to a given probability *p* (particularly the best one $a_{i,b}$ related to *p*=1, remaining, in principle, unknown) can be obtained by integration of the crack growth rate law, $da / dN = G(\Delta K)$, due to its uniqueness, for a generic number N_L of cycles to failure , i.e.:

$$\int_{a_{i,m}}^{a_{f,m}} \frac{da}{G(\Delta K(a))} = \int_0^{N_L} dN = N_L \quad \rightarrow \quad a_{f,m} = a_{f,m}(a_{i,m}, N_L, \Delta\sigma_m) \, . \tag{7}$$

Since Eq. (6) applies irrespective of the number of cycles considered, the above quantities can be used to write a system of two equations:

$$\frac{a_w}{a_m} = \frac{a_w(a_{i,w}, N, \Delta\sigma_w)}{a_m(a_{i,m}, N, \Delta\sigma_m)} = r \quad \text{and} \quad \frac{a_{f,w}}{a_{f,m}} = \frac{a_{f,w}(a_{i,w}, N_L, \Delta\sigma_w)}{a_{f,m}(a_{i,m}, N_L, \Delta\sigma_m)} = r \tag{8}$$

with $r = (\Delta\sigma_m / \Delta\sigma_w)^2$, the solution of which provides the quantities $a_{i,w}$ and $a_{i,m}$ that can be finally associated with the worst and a generic surface state of the material, respectively.

APPLICATIONS OF THE EXTENDED K-T DIAGRAM TO STEEL 1045

In order to demonstrate its utility for fatigue life assessment of mechanical and structural components, the above proposed approach is applied to the standard carbon steel 1045 taking into account the mechanical and fatigue properties as reported in Table 1 [6].

Steel 1045			***Chemical composition***		
	Carbon	Iron	Manganese	Phosphorus	Sulphur
	0.42 - 0.5	Balance	0.6 - 0.9	0.04 max	0.05 max
			Mechanical properties		
	E (GPa)	σ_y (MPa)	σ_u (MPa)	K	n
	202	382	621	4.0198E10	4.8077
			Fatigue properties		
$\Delta\sigma_0$ (MPa)	a_0 (m)	K_{fc}	K_{th}	*C'* (m/cycle)	*m*
275	2.1218E-4	80 MPa$\sqrt{m}$	7.1 MPa$\sqrt{m}$	8.20E-13	3.50

Table 1. Main chemical, mechanical and fatigue properties of the carbon steel 1045 (from [6]).

In the present case, rather than using the probabilistic model proposed by [3], the *S-N* field is defined by determining the percentile curves corresponding, respectively, to the probabilities of failure *p=0* and *p=0.90*, from the results of a previous testing program carried out for three different fatigue stress ranges ($\Delta\sigma = 300, 400, 500\, MPa$), as illustrated in Fig. 4.

The quantities $a_{worst} = \max(a_{\max})$ and $a_{best} = \min(a_{\max})$ are determined according to Eq. (6) by considering failure conditions related to $N = 10^5 cycles$, whereas, $r = (\Delta\sigma_b / \Delta\sigma_w)^2 = (455/340)^2 \cong 1.79$ (Fig. 4) or, equivalently, $c = \ln r \cong 0.583$.

By adopting the crack growth rate law by Donahue et al. [7], expressed by:

$$\frac{da}{dN} = G(\Delta K) = C \cdot (Y \cdot \Delta\sigma \cdot \sqrt{\pi \cdot a} - \Delta K_{th})^m \tag{9}$$

the crack growth law $a(N) = e^{F(N)}$ is assessed and used to get the expressions to be inserted into the system given by Eqs (8).

Since this system of equations is not easily solvable by analytical or numerical standard techniques, it has been tackled by using a genetic algorithm (GA) [8-9].

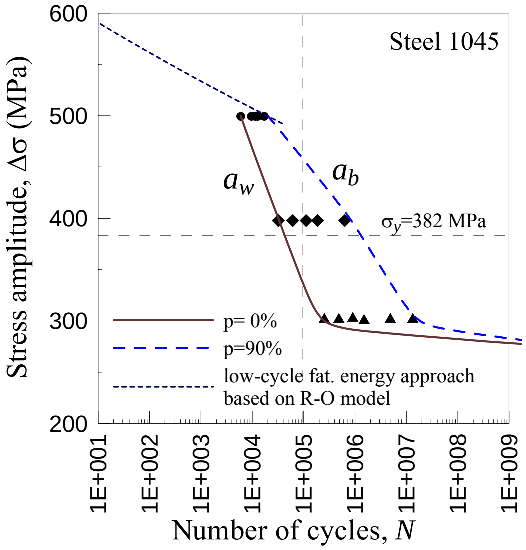
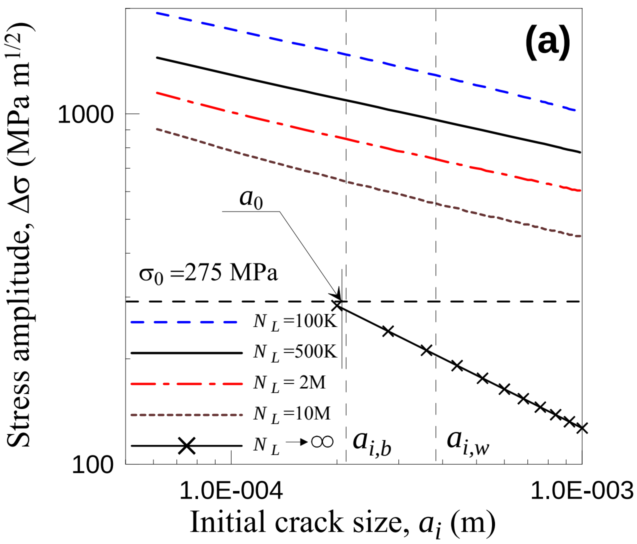


Figure 4. S-N curves corresponding to the p=0 and p=0.90 for the steel 1045.

The method operates by selecting, along the iteration process, the best solution among different possibilities, trying to fulfil as better as possible the required value of the objective function, here identified as the known ratio $r = (\Delta\sigma_b / \Delta\sigma_w)^2 \cong 1.79$ for both the initial and the final crack sizes.

After a short number of iterations, the sought values for $a_{i,b}$ and $a_{i,w}$ are obtained. The GA process produces a stable result for the two above quantities, namely $a_{i,b} = 2.12682 \cdot 10^{-4}$ m and $a_{i,w} = 3.80701 \cdot 10^{-4}$ m, irrespectively of the stress range and the final number of cycles N_L adopted (see Eq.(8)).



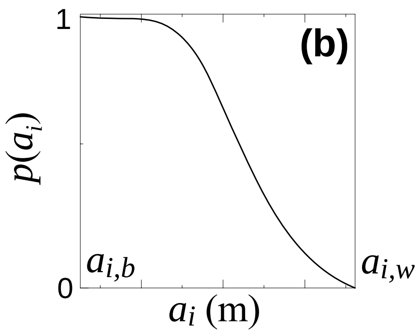


Figure 5. Kitagawa-Takahashi diagram for the steel 1045 for different values of cycles to failure N_L (a) with probability distribution related to the finishing surface state (b).

Finally, the generalised Kitagawa-Takahashi diagram has been obtained by integrating the Donahue equation starting from the initial crack size up to the fulfilment of the critical condition ($\Delta K_I(a_i, N_L) = \Delta K_{Ic}$) for a given number of cycles N_L to failure; the initial crack size a_i has been assumed to be simply uniformly distributed in the interval $a_{i,b}$ - $a_{i,w}$. As can be observed in Fig. 5, the initial crack size to be considered in the KT diagram corresponds to the probability of failure assumed, this being dependent on the finishing surface state.

CONCLUSIONS

The main conclusions to be drawn from the present study are the following:

- A probabilistic concept is proposed for being implemented in the Kitagawa-Takahashi-El Haddad diagram aiming at providing higher reliability in practical design cases.
- The approach allows us to establish a connection between the probabilistic S-N field and the crack growth rate curve, the latter being assumed for the present to be deterministic.
- The approach can be applied indistinctly either for an infinite or finite limit number of cycles.
- Further study is needed to allow an extension of the proposed approach to define the KT diagram for small cracks. i.e., in the LCF region, as well as the consideration of stochastic crack growth rate curves, particularly in the threshold regime.

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