Determination of the Stress Intensity Factor at the Single Edge Crack Tip Using RKPM

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Abstract

Reproducing Kernel Particle Method (RKPM) is a mesh-free technology which has proven very useful for solving problems of elastic-plastic fracture mechanics. In this study, the stress intensity factor (SIF) at the crack-tip in a work-hardening material is obtained using RKPM. Ramberg-Osgood stress-strain relation is assumed and the crack-tip SIF before and after formation of the plastic zone are examined. To impose the essential boundary conditions, penalty method is used. To construct the shape functions in the vicinity of the crack and crack-tip, both the diffraction and visibility criteria are employed and the crack tip region is also refined using more particles in two various model particle arrangements. The effects of different dilation parameters on SIF under plane-stress and plane-strain conditions are studied for plane-stress and plane-strain conditions. Results show that dilation parameter has a great impact on the performance of the RKPM and especially on the SIF value for the edge crack problems. The main objective is to study the effects of different dilation parameters on SIF value under plane-stress and plane-strain conditions at the crack-tip using diffraction and visibility criteria.

Keywords: Mesh-free, Reproducing Kernel Particle Method (RKPM), Crack-tip, Stress Intensity Factor (SIF), Dilation Parameter, Particle Arrangement

Introduction

Recently, mesh-free methods have been increasingly utilized in solving various types of boundary value problems. Mesh-free methods eliminate some or all of the traditional mesh-based view of the computational domain and rely on a particle view of the field problem. One of the oldest approaches in mesh-free methods is the Smooth Particle Hydrodynamics (SPH), which was first introduced in 1977 by Lucy Gingold and Monaghan [2]. SPH was first applied in astrophysics to model fluid dynamics phenomena. In 1993, Petschek [3] and Libersky extended SPH to solid mechanics. Recent advances on mesh-free methods are: element-free Galerkin method (EFGM) by Belytschko [1] at 1994, reproducing kernel particle method (RKPM) by Liu, et al. at 1996, and mesh-less local Petrov-Galerkin (MLPG) by Atluri [4] at 1999.

Mesh-free methods go back to the seventies. The major difference to finite element methods is that the domain of interest is discretized only with nodes, often called particles. In recent years, much research have been done on mesh-free methods for solving differential equation problems including crack and also obtained satisfactory results. Among these methods Reproducing Kernel Particle Method (RKPM) has been used increasingly in fracture mechanic problems. Boundary value problems (BVPs) often have essential boundary conditions (EBCs) that involve derivatives, for example, in beams and plates, where slopes are commonly enforced at the boundaries. Such problems are solved numerically using mesh-free techniques like the RKPM and the EFGM.

Throughout numerical analyses of fracture mechanics problems, the concept of shape function is crucial. The role of the shape functions is very important and decisive in numerical methods in which the approximation function of the system is replaced with the real function in the differential equation. Therefore, better and more accurate understanding of these functions and the effects of various parameters on their performance has significant impact on the effective analysis of different problems.

In 1968, Rice [5] presented the concept of energy release rate by means of *J*-integral. The *J*-integral represents a way to calculate the strain energy release rate, or work (energy) per unit fracture surface area, in a material. An important feature of the *J*-integral is that it is path independent and it helps to calculate the *J*-integral at a far distance from the crack tip. In linear elastic fracture mechanics the *J*-integral has a direct relationship with the stress intensity factors (SIFs). In this study the *J*-integral has been used to calculate the SIF at the crack tip.

Review of Reproducing Kernel Particle Method

SPH method first was introduced in 1977 by Lucy Gingold and Monaghan [2]. In the SPH method, system response is reproduced by invoking the notion of a kernel approximation for *f(x)* on domain Ω. This method is not accurate on the boundary conditions, or when few particles are considered on the domain unless the lumped volume is carefully selected. RKPM is an alternative method to formulate the discrete consistency that is lacking in the SPH method. The foundation of the RKPM was proposed by Liu et al. [7] in 1993 and applied to computational mechanics. RKPM modifies the kernel function by introducing a correction function $C(\xi;\xi-x)$. Adding the correction function in the kernel approximation significantly enhances the solution accuracy in comparison to the SPH method. The method of using corrected kernel approximation in reproducing a function is called Reproducing Kernel Particle Method. The reproduced kernel function of u(x) can be written as Equation1:

$$u^R(\xi) = \int_{\Omega} u(x)\bar{\phi}(\xi;\xi-x)dx \tag{1}$$

where $\bar{\phi}(\xi;\xi-x)$ is the modified kernel function on domain *Ω* that is expressed by Equation2:

$$\bar{\phi}(\xi;\xi-x) = C(\xi;\xi-x)\phi(\xi-x) \tag{2}$$

$$\phi_a(\xi-x_i) = \frac{1}{a}\phi\left(\frac{\xi-x_i}{a}\right) \tag{3}$$

where $\phi_a(\xi-x)$ is window function, $C(\xi;\xi-x)$ is a correction function, and *a* is the dilation parameter of the kernel function. Dilation parameter is defined in order to make more flexibility for the window function and this parameter will control the expansion of the window function on the domain. The correction function $C(\xi;\xi-x)$ proposed by Liu et al. is shown by a linear combination of polynomial including some unknown coefficients. These unknown coefficients will be computed after imposing the boundary conditions. In order to get the equations for reproducing an arbitrary function, consider the following Taylor series expansion:

$$u(x)=\sum_{\alpha=0}^{\infty}\frac{(-1)^{\alpha}}{\alpha!}(\xi-x)^{\alpha}u^{(\alpha)}(\xi) \tag{4}$$

Substituting Equation4 into Equation1 leads to:

$$u^{R}(\xi)=\sum_{\alpha=0}^{\infty}\frac{(-1)^{\alpha}}{\alpha!}\left(\int_{\Omega}(\xi-x)^{\alpha}\bar{\phi}(\xi;\xi-x)dx\right)u^{(\alpha)}(\xi) \tag{5}$$

In order to simplify Equation5, the α^{th} degree moment matrix of function $\bar{\phi}_a(\xi;\xi\text{-}x)$ is defined by:

$$\bar{m}_{\alpha}(\xi)=\int_{\Omega}(\xi-x)^{\alpha}\bar{\phi}_{a}(\xi;\xi-x)dx \tag{6}$$

Then Equation5 will be rewritten in the form of Equation6:

$$u^{R}(\xi)=\overline{m_{0}}(\xi)u(\xi)+\sum_{\alpha=1}^{\infty}\frac{(-1)^{\alpha}}{\alpha!}\overline{m_{\alpha}}(\xi)u^{(\alpha)}(\xi) \tag{7}$$

In order to exactly reproduce the nth order polynomial function, the following conditions need to be satisfied;

$$\begin{cases}\bar{m}_{o}(\xi)=1 & \\ \bar{m}_{\alpha}(\xi)=0 & \alpha=1,2,...,n\end{cases} \tag{8}$$

Or in summary:

$$\bar{m}_{\alpha}(\xi)=\delta_{\alpha o} \quad ; \quad \alpha=0,1,2,...,n \tag{9}$$

If a correction function including n+1 unknown coefficient is defined, n+1 Equations of 9 can be satisfied simultaneously. The correction function is defined by Equation10:

$$C(\xi,\xi-x)=\sum_{\alpha=0}^{n}\beta_{\alpha}(\xi)(\xi-x)^{\alpha} \tag{10}$$

It can be also expressed in matrix form:

$$C(\xi;\xi-x)=P^{T}(\xi-x)\beta(\xi) \tag{11}$$

where $P^T(\xi - x)$ is a set of basic functions and including $n+1$ components and $\beta(\xi)$ is a set of unknown coefficient. Substituting Equation11 into Equation9 and considering definition of moment matrix in Equation6 leads to Equation13:

$$\int_{\Omega}\left\langle(\xi-x)^{\alpha}(\xi-x)^{\alpha+1}...(\xi-x)^{\alpha+n}\right\rangle\phi_{a}(\xi-x)dx \tag{12}$$

$$\begin{Bmatrix}\beta_{o}(\xi)\\ \beta_{1}(\xi)\\ \vdots\\ \beta_{n}(\xi)\end{Bmatrix}=\delta_{\alpha o} \qquad \alpha=1,2,...,n \tag{13}$$

From Equation13 the unknown coefficient sets of $\beta_i(\xi)$ are obtained. Equation13 can also be rewritten as Equation14.

$$\left\langle m_{\alpha}(\xi) \quad m_{\alpha+1}(\xi) \quad ... \quad m_{\alpha+n}(\xi)\right\rangle\begin{Bmatrix}\beta_{o}(\xi)\\ \beta_{1}(\xi)\\ \vdots\\ \beta_{n}(\xi)\end{Bmatrix}=\delta_{o\alpha} \tag{14}$$

Or it can be shown in matrix form as Equations 15:

$$M(\xi)\beta(\xi) = P(0) \tag{15}$$

Moment matrix M can be shown as Equation 16:

$$M(\xi) = \int_{\Omega} P(\xi - x)P^T(\xi - x)\phi_a(\xi - x)dx \tag{16}$$

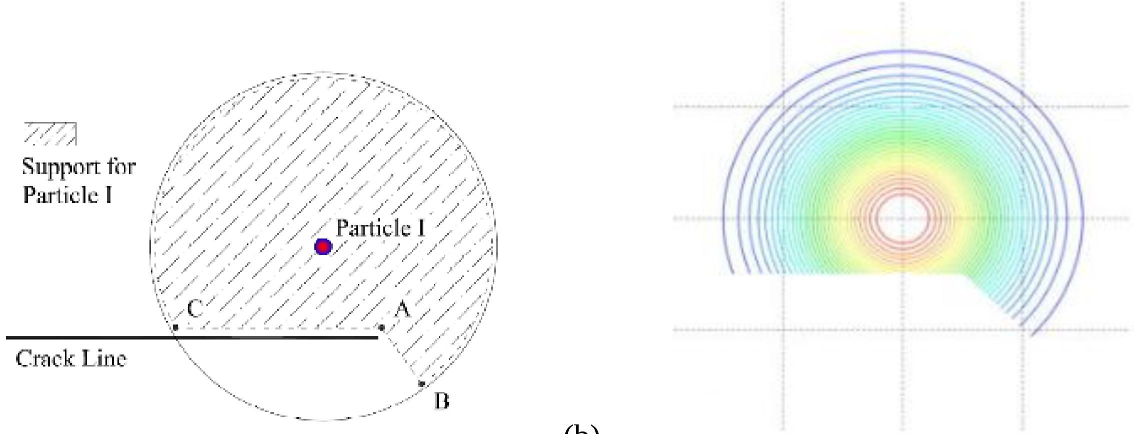
Since the window function is always positive, all the components of moment matrix are linearly independent with respect to ϕ_a. Therefore, the moment matrix is nonsingular. Hence, simultaneously solving Equation15, the unknown coefficient sets of $\beta_i(\xi)$ are obtained:

$$\beta(\xi) = M^{-1}(\xi)P(0) \tag{17}$$

After obtaining the unknown coefficient sets $\beta_i(\xi)$ the correction function can be easily calculated from Equation10.

Modification of RKPM Shape Functions

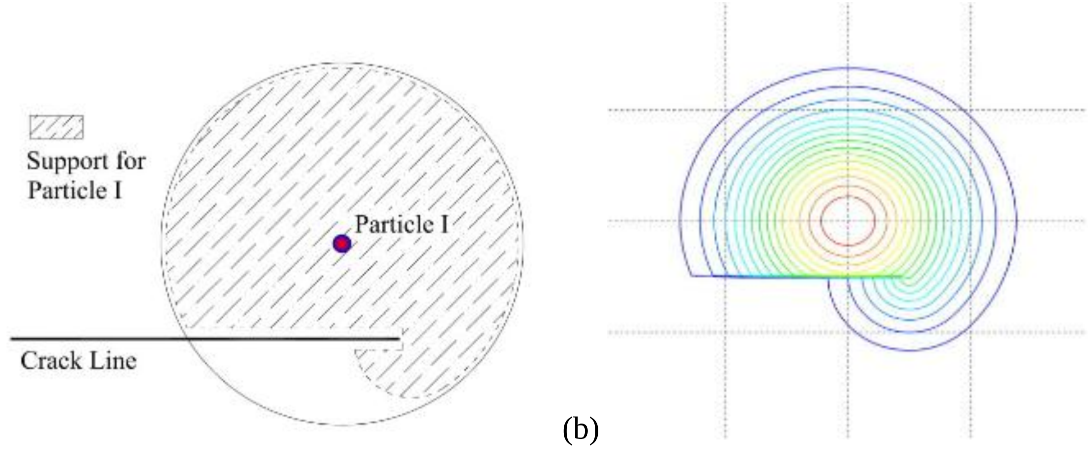
Through engineering problems, the domain of the problem may contain non-convex boundaries, particularly the fracture ones having discontinuous displacement fields. In such conditions, the shape functions associated with particles, whose supports intersect the discontinuity, should be modified. One of these criteria is the visibility introduced by Belytschko, Lu, Gu [1] (1994) and Krysl and Belytschko [14] (1996). In this approach, if the assumed light beam meets the discontinuity line, the shape function after the barrier will be cut. Therefore, a discontinuity is applied to the geometry. For example, if a crack is considered, the influence domain of particles I and J close to the crack tip using visibility criterion can be shown as Figure 1a. As can be seen, the particles that at particle I or J cannot be seen by an observer will be removed. In the other words, the window function and shape function of the particles which the crack or discontinuity prevent from reaching the light beam will be modified to amount to a zero as shown in Figure 1b.



(a) (b)

Figure 1. (a) Modified Influence Domain (b) Modified Window Function Contour of the Particles Next to the Line of Discontinuity Using Visibility Criterion

Diffraction criterion (*Organ* and *Belytschko* [1] in 1996) is based on the bending of the light beam which has been described in the visibility criterion around a tip discontinuity. Consider the end of the discontinuity line in Figure 3. If the distance between the crack tip and the end of the arc is called d for particle I, then a circle with the center being the crack tip and radius of d is drawn. Areas outside the circle and behind the discontinuity are removed and the amount of the shape function in these areas will be zero (Figure 2).



(a) (b)

Figure 2. (a) Modified Influence Domain (b) Modified Window Function Contour of the Particles Next to the Line of Discontinuity Using Diffraction Criterion

Stress Intensity Factor

The main purpose of fracture mechanics is to determine the status of cracks in different loading conditions. Stress, strain, displacement, and energy fields are required to obtain a driving force for crack growth. SIF and *J*-integral are two important concepts of crack problems. SIF is used to quantify the stress field around the crack tip. Many methods have been developed to determine the stress intensity factor. One of these methods to calculate the stress intensity factor is *J*-integral. If a node is considered with distance *r* and angle of α with the x-axis in the vicinity of the crack edge, then the stress field in this node is calculated according to the Irwin method in different crack modes. Therefore, stress field in the crack tip for linear elastic materials is calculated by Equation 18:

$$\sigma_{ij} = \frac{K}{\sqrt{2\pi r}} f_{ij}(\theta) \tag{18}$$

K parameter is the SIF for different modes in the crack tip, and shown K_I, K_{II}, and K_{III} are for the first, second and third mode. Values of these coefficients are determined according to the dimensions and loading condition of the problem. Therefore, the SIF relationship is calculated from the analysis of the geometrical and loading condition. K_I, K_{II}, and K_{III} are physically the intensity of force transfer at the crack tip due to creation of the crack in the material. SIF plays an important role as a failure parameter. Rice (1968) also showed that this integral has linear elastic attitude with the energy release rate and was independent of the path around a crack. The two-dimensional *J*-integral was defined as Equation 19:

$$J = \int_{\Gamma} \left(W dx_2 - \sigma_{ij} n_j u_{i,1} ds \right) \qquad j = 1,2 \tag{19}$$

where *W* is strain energy density, *σ* is stress tensor, *n* is the normal to the curve *Γ*, and *u* is the displacement vector. The strain energy density is given by:

$$W = \int_0^{\varepsilon_{ij}} \sigma_{ij} d\varepsilon_{ij} \tag{20}$$

Also, *J*-integral can be obtained in terms of SIF of the first, second, and third mode.

$$J = \frac{1}{E'}\left(K_I^2 - K_{II}^2\right) + \frac{1}{2\mu} K_{III}^2$$

$$E' = \begin{cases} \dfrac{E}{1-\nu^2} & Plane \quad Strain \\ E & Plane \quad Stress \end{cases} \tag{21}$$

where μ is shear modulus, E' is modulus of elasticity, and ν is poisson ratio. An important feature of the *J*-integral is that it is path independent and this helps to calculate the *J*-integral in a far distance from the crack tip. Then SIF is calculated from Equation 22 for plane-stress and plane-strain conditions:

$$
\begin{aligned}
&J = \frac{K^2}{E} \\
&K = \sqrt{JE} \quad Plane - stress \\
&J = \frac{K^2}{E}\left(1-\nu^2\right) \\
&K = \sqrt{\frac{JE}{1-\nu^2}} \quad Plane - strain
\end{aligned}
\tag{22}
$$

Edge Crack Modeling in RKPM

With what was stated in previous, and using a FORTRAN program that was written for solving the liner elastic on a steel plate with specified dimension using RKPM. The stress, strain, and displacement field in x and y direction in all computational particles and calculation of SIF under plane-stress and plane-strain conditions were obtained. Penalty method is used to apply the boundary conditions. Penalty coefficient, β, is adopted as 10^6 *E*, in which *E* is Young's modulus. A rectangular steel plate is selected with dimensions of 2×1 m^2. An edge crack is considered with a length of 0.2 m in the middle of the plate. A tensile stress of 150 MPa is applied at the bottom and the top of the plate. The loading increment is assumed 10 MPa. Roller constraint is used for the plane in front of the crack and pin constraint is used for the front face of the plate (Figure 3).

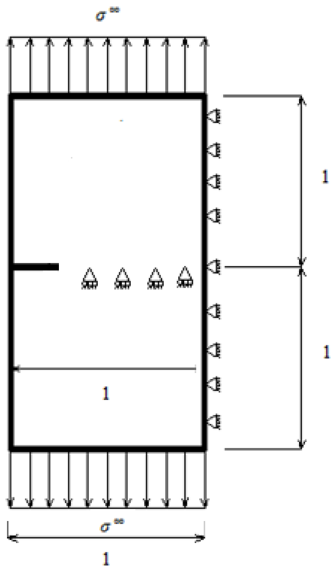


Figure 3. Domain and Boundary Conditions

Spline 3rd degree is used as a window function. The modulus of elasticity of the plate is 207,000MPa, Poisson ratio of 0.3 and hardening parameter *n*=10. The problem is investigated in three different conditions: (1) 800 particles uniformly scattered on the surface of the plate, and 28 particles positioned on the circles with angles of 45 degree around the crack tip (Figure 4), (2) 800 particles uniformly scattered on the surface of the plate, and 60 particles are positioned on the circles with angles of 22.5 degree around the crack tip (Figure 5).

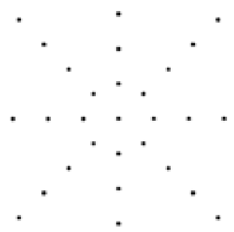


Figure 4. Star Arrangement (a-model)

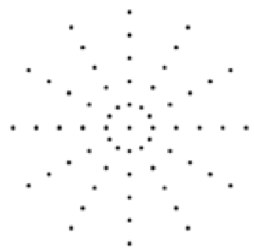


Figure 5. Circle Arrangement (b-model)

For this plate, dilation parameters are compared for two visibility and diffraction criteria. Two different arrangements: a-model and b-model arrangements are considered for the particles as shown in Figures 4 and 5. The Gaussian number is considered to be equal to 3. Figures 6 to 9 show SIF versus dilation parameter in elastic and plastic conditions. It can be concluded that b-model arrangement using diffraction criterion has the better results. In comparing between the visibility and diffraction methods to modify the shape functions, the diffraction criterion seems to have better results for the SIF in both the elastic and plastic analysis.

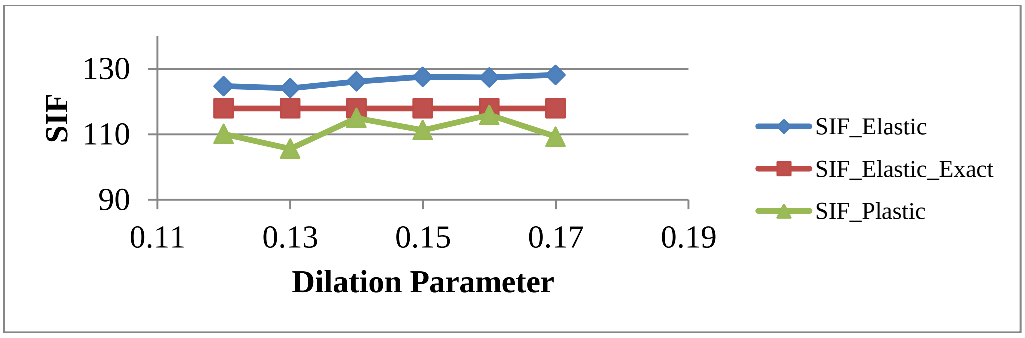


Figure 6. SIF vs. Dilation Parameter in a-model Particle Arrangement Using Diffraction Criterion

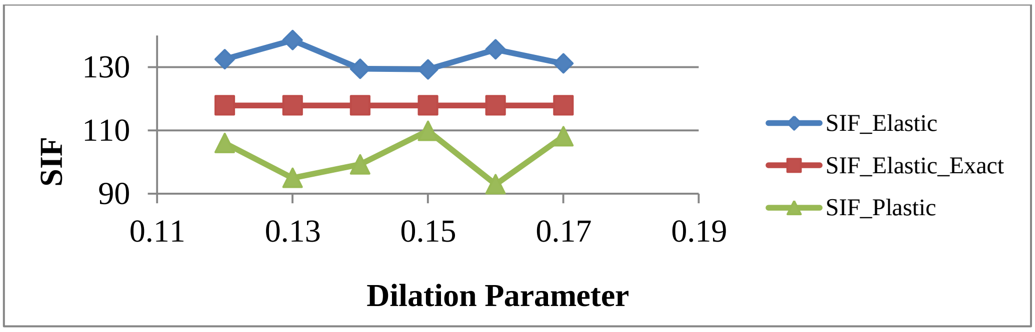


Figure 7. SIF vs. Dilation Parameter in a-model Particle Arrangement Using Visibility Criterion

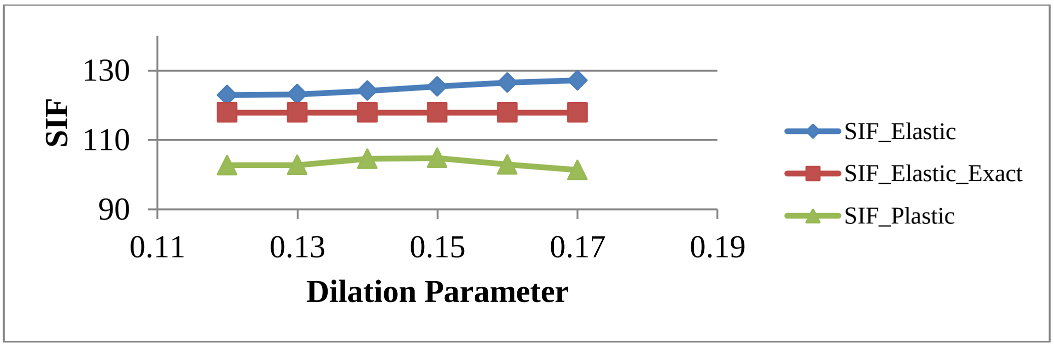


Figure 8. SIF vs. Dilation Parameter in b-model Particle Arrangement Using Diffraction Criterion

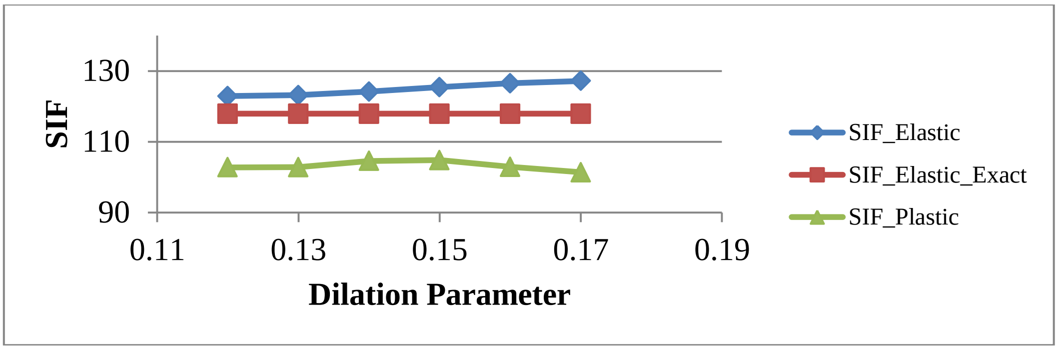


Figure 9. SIF vs. Dilation Parameter in b-model Particle Arrangement Using Visibility Criterion

The FORTRAN program developed for the elastic-plastic material is able to recognize the yielded particles in according to Von-Mises criterion. The crack tip region was refined using more particles in the a-model and b-model particle arrangements. For the same plate with dimensions 2×1 including 828 particles, and 860 particles SIF graphs are shown for plane strain and stress conditions. Tensile stress of 150 MPa is applied at the bottom and the top of the plate. In each 10 MPa loading increment, the SIF values are calculated. Figure 10 shows the SIF values versus tensile stress for plane-stress and plane-strain conditions.

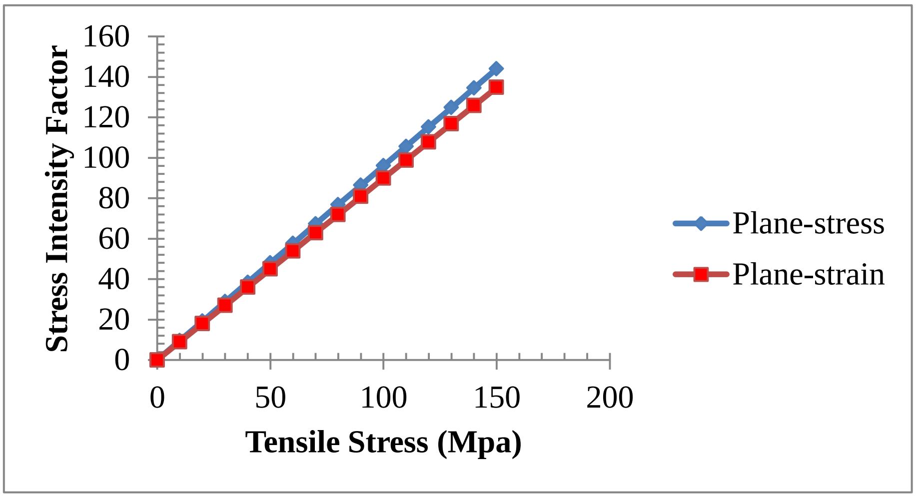


Figure10. SIF Values vs. Tensile Stress

Conclusions

1) When the dilation parameter increases for all particles in plastic analysis, the difference of the *J*-integral in fully plastic regions and fully elastic regions will increase. The reason for this is that when the crack is analyzed in the elastic-plastic condition, for the particles in crack tip with less dilation parameter *J*-integral is calculated in fully plastic and domain of influence domain does not enter to elastic region. Also, the SIF versus dilation parameter graphs show that increasing the density of particles at the crack tip using b-model particle arrangement will result in more realistic answers for SIF.

2) In comparing between the visibility and diffraction methods to modify the shape functions, the Diffraction criterion seems to have better results for the SIF in both the elastic and plastic analysis.

3) Stress Intensity Factor at the crack tip for the plane-stress condition is bigger than that in the plane-strain condition. The reason for this is due to limitations in the third dimension for the plane-strain condition (Figure 10).

References

1. Belytschko T., Y. Krongauz, D. Organ, M. Fleming, P. Krysl. Meshless Methods (1996): An overview and recent developments Computer methods in applied mechanics and engineering, Vol 139, pp3-47.
2. Jin, X.; Li, G.; Aluru, N. R. (2001): “On the equivalence between least-squares and kernel approximations in meshless methods.” *Computer Modeling in Engineering and Sciences*, vol. 2, pp. 447-462.
3. Nguyen, V. P.; Rabczuk, T.; Bordas, S.; Duflot, M. (2008): “Meshless methods: A review and computer implementation aspects.” *Mathematics and Computers in Simulation*, vol. 79, pp. 763-813.
4. Zhu, T.; Atluri S. N. (1998): A modified collocation method and a penalty formulation for enforcing the essential boundary conditions in the element free Galerkin method. *Computational Mechanics*, vol. 21, pp. 211-222.
5. Rice, J. R. (1968), “A path independent integral and the approximate analysis of strain concentration by notches and cracks.” *Journal of Applied Mech.*, vol. 35, pp. 379-386.
6. Atluri, S. N.; Shen, S. (2002a): The meshless local Petrov-Galerkin (MLPG) method. Tech. Science Press, 440 pages.
7. Liu, W. K., Jun, S., Zhang, Y. F. (1995): “Reproducing Kernel Particles Methods.” *International Journal for Numerical Methods in Fluids*, vol. 20, pp. 1081-1106.
8. Atluri, S. N.; Zhu, T. (1998): A new meshless local Petrov-Galerkin (MLPG) approach in computational mechanics. *Computational Mech.*, Vol. 22, pp. 117-127.
9. Nguyen, V. P.; Rabczuk, T.; Bordas, S.; Duflot, M. (2008): “Meshless methods: A review and computer implementation aspects.” *Mathematics and Computers in Simulation*, vol. 79, pp. 763-813.
10. Rao, B. N.; Rahman, S. (2004): “An enriched meshless method for non-linear fracture mechanics.” *Int. Journal for Numerical Methods in Engineering*, vol. 59, pp. 197-223.
11. Niederegger, P., Knobloch, M., Fontana, M., (2006): “Elements with nonlinear stress-strain relationships subjected to local buckling.” *Taylor & Francis Group, London*, ISBN 0-415-40817-2.
12. Atluri, S. N. (2004): “The Meshless Local Petrov-Galerkin (MLPG) Method for Domain&Boundary Discretizations, Tech Science Press, 665 pages.
13. Atluri, S. N.; Han, Z. D.; Shen, S. (2003): Meshless Local Patrov-Galerkin (MLPG) approaches for weakly singular traction & displacement boundary integral equations, *CMES: Computer Modeling in Engineering&Science*, vol. 4, no. 5, pp. 507-517.
14. Krysl, P., and Belytschko, T., "Element-Free Galerkin Method: Convergence of the Continuous and Discontinuous Shape Functions," Computer Methods in Applied Mechanics and Engineering Volume 148, 257-277 (1997).