Crack paths near the interface between anisotropic solids

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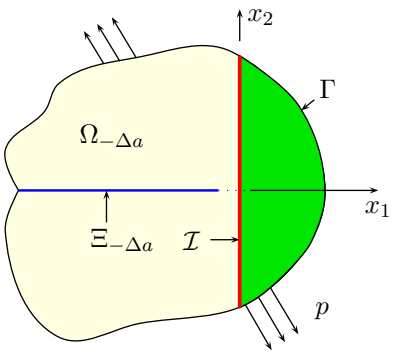
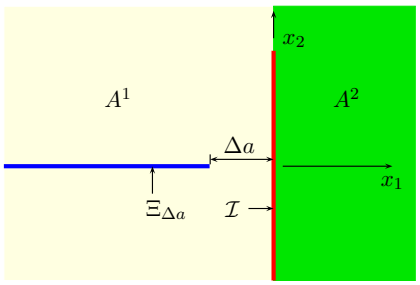
ABSTRACT. *If a crack approaches the interface between two dissimilar anisotropic materials various scenarios can happen. The question whether the crack will reach or even penetrate the interface depends on the mismatch of elastic moduli in the two materials. This contribution is devoted to the question whether a crack will reach the interface when the distance of the crack tip and the interface are small compared to the distance of the crack tip to the outer boundary. The energy release is calculated using the method of matched asymptotic expansions. Other than for the calculation of the ERR in homogeneous materials here the reference problem is the situation when the crack has already reached the interface.*

INTRODUCTION

The application of anisotropic composite materials in modern engineering leads to new challenges in fracture mechanics. If a crack approaches an interface between two different anisotropic materials experiments show that the crack can stay stuck at the interface, it may pass through the interface or be deflected.

In this contribution we consider a straight crack starting from the boundary in a body composed of two dissimilar anisotropic brittle materials as indicated in the figures. In particular we use energy arguments to address the following problem: Suppose the the crack tip is located in a small distance Δa from the interface, is it possible that the crack propagates and reaches the interface? In order to do so we must calculate the energy release rate if the crack tip moves from the point $(-\Delta a, 0)$ to the point $(0, 0)$. To be more specific, we consider a plane elasticity problem: Let Ω be a domain in the plane $\mathbb{R}^2$ with boundary Γ, the closure $\overline{\Omega}$ represents a body composed of two materials with related Hooke tensors $\mathbf{A}^1$ and $\mathbf{A}^2$, respectively.





The interface $\mathcal{I}$ is located on the x_2-axis, while the crack $\Xi_{-\Delta a}$ is located on the x_1 axis:

$$\Xi_{-\Delta a} = \{(x_1, 0) \in \Omega, x_1 \leq -\Delta a\},$$

in particular Ξ_0 corresponds to the crack with tip at the interface $\mathcal{I}$. Let $u^{-\Delta a}$ be the displacement field solving the elasticity problem,

$$\begin{aligned} -\nabla \cdot \boldsymbol{\sigma} &= 0 \qquad \text{in } \Omega_{-\Delta a} = \Omega \setminus \Xi_{-\Delta a}, \\ \sigma_{12} = \sigma_{22} &= 0 \quad \text{on } \Xi_{-\Delta a,\pm}, \quad \boldsymbol{\sigma} \cdot n = p \quad \text{on } \Gamma, \end{aligned} \tag{1}$$

where n denotes the external unit normal vector. We have Hooke's law

$$\boldsymbol{\sigma} = \mathbf{A}(x) \cdot \boldsymbol{\varepsilon}, \qquad \mathbf{A}(x) = \begin{cases} \mathbf{A}^1, & x_1 < 0 \\ \mathbf{A}^2, & x_1 > 0. \end{cases} \tag{2}$$

Furthermore, we have to assume that the external loading p is self balanced, that is $\int_\Gamma p_i = 0$, $i = 1, 2$, $\int_\Gamma p_1 x_2 - p_2 x_1 ds = 0$. In addition we require the continuity of the displacement fields and normal stresses at the interface $\mathcal{I}$: on the interface $\mathcal{I}$:

$$u(0_+, x_2) = u(0_-, x_2), \quad \sigma_{i1}(0_+, x_2) = \sigma_{i1}(0_-, x_2). \tag{3}$$

The associated potential energy $\mathbf{U}$ can be considered as a function of the distance Δa (we use the sum convention):

$$\mathbf{U}(-\Delta a) = \frac{1}{2} \int_{\Omega_{-\Delta a}} \sigma_{ij} \varepsilon_{ij} - \int_\Gamma p_j u_j ds.$$

In order to decide whether the crack can reach and eventually penetrate the interface, the following condition must be fulfilled for all (small) values of Δa:

$$-\Delta \mathbf{U} = \mathbf{U}(-\Delta a) - \mathbf{U}(0) > 2\gamma_0 \Delta a, \tag{4}$$

where $2\gamma_1 \Delta a$ represents the energy to produce the new surface, that is $2\gamma_1$ is the critical energy release rate in the first material. Problems of this type were studied

for isotropic materials for example in [1], [2].

POWER LAW SOLUTIONS AND WEIGHT FUNCTIONS

Usually the energy release rate is expressed in terms of the stress intensity factors related to the starting point of the crack growth which would be the point $(-\Delta a, 0)$ here. However, if Δa is small it is more adequate to use the domain Ω_0 with crack tip on the interface $\mathcal{I}$ as a reference configuration. Then the change $\Delta\mathbf{U}$ of the potential energy is expressed with coefficients of the near field expansion of the displacement field u^0 around the tip $(0,0)$. To this end we need the *power law solutions* X^i, Y^i (e.g. [3, 4]) to the elasticity problem in composites, which are but solutions to the elasticity problem in the plane with semi-infinite crack $\Xi_{\infty,\pm} = \{(r,\varphi) : \varphi = \pm\pi\}$, ($r, \varphi$ polar coordinates around the crack tip $(0,0)$):

$$-\operatorname{div}\boldsymbol{\sigma} = 0 \text{ in } \mathbb{R}^2 \setminus \Xi_{\infty,\pm}, \quad \sigma_{12} = \sigma_{22} = 0 \text{ on } \Xi_{\infty,\pm}, \tag{5}$$

completed with Hooke's law (2) and the transmission condition (3). There are always two sequences of power-law solutions

$$X^j(r,\varphi) = r^{\lambda_j}\Phi^j(\varphi), \operatorname{Re}\lambda_j > 0, \qquad Y^j(r,\varphi) = r^{-\lambda_j}\Psi^j(\varphi), \tag{6}$$

the numbers λ_j being *generalized eigenvalues*. For our purpose it is enough to consider only the values close to zero. For homogenous solids that is $\mathbf{A}^0 = \mathbf{A}^1$ it is well known [5, 6] that $\lambda = \pm 1/2$ are double eigenvalues with related pairs of power-law solutions of the form (6). If the Hooke tensors are different and in particular related to anisotropies the following situations may occur as perturbations of the eigenvalue $\lambda = 1/2$:

Case 1: Two simple real eigenvalues $0 < \lambda_1 \leq \lambda_2 < 1$ with $X^i = r^{\lambda_i}\Phi^i(\varphi)$,

Case 2: A pair of conjugate complex eigenvalues $\lambda_1 = \lambda$, $\lambda_2 = \overline{\lambda}$ with

$$X^1 = X = r^{\lambda}\Phi(\varphi), \qquad X^2 = \overline{X} = r^{\overline{\lambda}}\,\overline{\Phi}(\varphi).$$

Case 3: A double real eigenvalue $0 < \lambda < 1$ with

$$X^1 = r^{\lambda}\Phi^1(\varphi), \quad X^2 = r^{\lambda}\ln r\Phi^1(\varphi) + r^{\lambda}\Phi^2(\varphi).$$

Case I appears always if the two materials are isotropic [7], the eigenvalues are found as roots of a transcendental equation. In [8] conditions on the elastic moduli were derived under which case 2 or 3 happen. The displacement field u^0 has the near field decomposition near the tip $(0,0)$:

$$u^0 \sim K_1X^1 + K_2X^2 + \ldots, \tag{7}$$

for a complex eigenvalue λ this simplifies to

$$u^0 \sim 2\,\mathrm{Re}(KX) + \ldots = 2(\mathrm{Re}\,K\,\mathrm{Re}\,X - \mathrm{Im}\,K\,\mathrm{Im}\,X) + \ldots$$

For each pair X^1, X^2 of power-law solutions there exists a pair of dual power-law solutions

$$Y^i = r^{-\lambda_i}\Psi^i(\varphi),\ i = 1, 2 \qquad \text{Case 1} \tag{8}$$

$$Y^1(x) = r^{-\overline{\lambda}}\,\overline{\Psi(\varphi)}, \quad Y^2(x) = r^{-\lambda}\Psi(\varphi) \qquad \text{Case 2} \tag{9}$$

$$Y^1(x) = r^{-\lambda}\Psi^1(\varphi) - r^{-\lambda}\ln r\Psi^2(\varphi), Y^2(x) = r^{-\lambda}\Psi^2(\varphi). \qquad \text{Case 3} \tag{10}$$

By Clayperon's theorem the potential energy can be represented as an integral over the external boundary:

$$\mathbf{U}(-\Delta a) = -\frac{1}{2}\int_\Gamma p \cdot u^{-\Delta a}\,ds, \text{ hence } \Delta\mathbf{U} = \frac{1}{2}\int_\Gamma p \cdot (u^{-\Delta a} - u^0)\,ds, \tag{11}$$

which means we can calculate the energy release if we know the difference $u^{-\Delta a} - u^0$ on the external boundary Γ. To calculate this as least asymptotically we use the method of matched asymptotic expansions [9, 10, 11]. For small Δa, near the external boundary Γ, the solution $u^{-\Delta a}$ will not differ too much from the solution u^0, hence $u^{-\Delta a}$ is approximated by an *outer expansion*

$$u^{-\Delta a}(x) \sim u^0(x) + a_1\zeta^1(x) + a_2\zeta^2(x) + \ldots, \qquad |x| \gg \Delta a. \tag{12}$$

The functions ζ^j are so-called weighting functions (cf. [12],[13]),

$$\zeta^j = Y^j + \widetilde{\zeta}^j$$

here $\widetilde{\zeta}^j$ are solutions to the problem (1), (3) for $\Delta a = 0$ and $p = -\boldsymbol{\sigma}(Y^j)$, hence ζ^j are singular at the crack tip $(0, 0)$, moreover

$$\zeta^j \sim Y^j + m_{jk}X^k + \ldots, \quad |x| \to 0. \tag{13}$$

Depending on the fixing of X^j, the dual power law solution Y^j can always be normalized in such a way that

$$K_j = \int_\Gamma p \cdot \zeta^j ds, \quad \text{resp.} \quad K = \int_\Gamma p \cdot \overline{\zeta} ds, \tag{14}$$

which plugged into (11) gives

$$\begin{aligned} 2\Delta\mathbf{U} &= K_1 a_1(\Delta a) + K_2 a_2(\Delta a) + \ldots, \text{ or} \\ \Delta\mathbf{U} &= \mathrm{Re}(K\overline{a}_1(\Delta a)) + \ldots, \end{aligned} \tag{15}$$

respectively, observe that in Case 2 we have $a_2 = \overline{a_1}$. To determine the coefficients a we use the *i*nner decomposition of $u^{-\Delta a}$. Passing to the stretched coordinates

$\xi = x/\Delta a$ fixes the crack tip in the point $(-1, 0)$ while the outer boundary moves to infinity as $\Delta a \to 0$. We define the weight functions at infinity, $\eta^j = X^j + \widetilde{\eta^j}$, as solution to the elasticity problem (in ξ-coordinates) in the plane with half infinite crack ending in $(-1, 0)$, the interface is situated on the line $x_1 = 0$, and $\widetilde{\eta^j}$ is regular at infinity, that is

$$\widetilde{\eta^j}(\xi) \sim M_{jk} Y^k(\xi) + \dots, \quad |\xi| \to \infty. \tag{16}$$

The 2×2 matrix $\mathbf{M} = (M_{jk})$ is symmetric resp. hermitian and *negative definite*. Near the crack tip, the solution $u^{-\Delta a}$ is approximated in terms of solutions of these solutions in stretched coordinates, that is by the *inner expansion* of the form

$$u^{\Delta a}\big(\Delta a^{-1} x\big) = u^{\Delta a}(\xi) \sim b_1 \eta^1(\xi) + b_2 \eta^2(\xi) + \dots, \qquad \Delta a \ll 1. \tag{17}$$

The coefficients a_j in (12) and b_j in (17) depend on the distance Δa, of course.

THE CALCULATION OF THE ENERGY RELEASE RATES

Exploiting the homogeneity relations of the power-law solutions X^j, Y^j together with the representation (16) of the weight functions η^j, the asymptotic representation at infinity of the inner decomposition (17) can be rewritten in x coordinates. The asymptotic representation (13) of the weight functions ζ^j gives the asymptotics of the inner decomposition (17) near the point $(0, 0)$. In a matching zone between the crack tip and the outer boundary Γ both decompositions must coincide. Equalizing the coefficients in front of the power-law solutions X^j, Y^j gives a system of four equations for the coefficients a_j and b_j, $j = 1, 2$. To be more specific, we put

$$\mathbf{K} = \begin{pmatrix} K_1 \\ K_2 \end{pmatrix}, \qquad \mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix},$$
$$\mathbf{m} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}, \qquad \mathbf{M} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix},$$

where K_j are the coefficients in eq. (7), m_{ij} are given in eq. (13), and M_{ij} in eq. (16).

Case 1: Two real eigenvalues $0 < \lambda_1 \le \lambda_2$**.**
With

$$\mathbf{M}(\Delta a) = \begin{pmatrix} (\Delta a)^{\lambda_1} & 0 \\ 0 & (\Delta a)^{\lambda_2} \end{pmatrix} \cdot \mathbf{M} \cdot \begin{pmatrix} (\Delta a)^{\lambda_1} & 0 \\ 0 & (\Delta a)^{\lambda_2} \end{pmatrix}$$

we obtain

$$\mathbf{a} = \mathbf{M}(\Delta a) \cdot \big(\mathbb{I} - \mathbf{m} \cdot \mathbf{M}(\Delta a)\big)^{-1} \cdot \mathbf{K}, \quad \mathbb{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{18}$$

It is clear that the inverse matrix appearing in this formula exist for small Δa. Using the Neumann expansion we find

$$\big(\mathbb{I} - \mathbf{m}\cdot\mathbf{M}(\Delta a)\big)^{-1} = \mathbb{I} + \mathbf{m}\cdot\mathbf{M}(\Delta a) + O(\Delta a)^2,$$

which gives

$$\mathbf{a} = \mathbf{M}(\Delta a)\cdot\mathbf{K} + O(\Delta a)^2.$$

By eq. (15) the main term in the energy release can be calculated to

$$\Delta\mathbf{U} = \frac{1}{2}\left((\Delta a)^{2\lambda_1} M_{11}(K_1)^2 + 2(\Delta a)^{\lambda_1+\lambda_2} M_{12}K_1K_2 + (\Delta a)^{2\lambda_2} M_{22}(K_2)^2 + \ldots\right) \tag{19}$$

Case 2: A pair of complex eigenvalues $\lambda_1 = \lambda$, $\lambda_2 = \overline{\lambda}$.
Here the weight functions also come in complex conjugate pairs

$$\zeta^1 = \zeta, \quad \zeta^2 = \overline{\zeta}, \qquad \eta^1 = \eta, \quad \eta^2 = \overline{\eta}.$$

while for the complex intensity factor K we have to take eq. $(14)_2$. The matrices $\mathbf{m}$ and $\mathbf{M}$ have complex entries now as well, but still are hermitian, and $\mathbf{M}$ is negative definite. In particular, from (16) it follows that $\mathbf{M}$ can be written in the form

$$\mathbf{M} = \begin{pmatrix} M_1 & M_2 \\ \overline{M_2} & M_1 \end{pmatrix} \tag{20}$$

The homogeneity relations for the X and Y now lead to

$$\mathbf{M}(\Delta a) = \begin{pmatrix} (\Delta a)^{\lambda} & 0 \\ 0 & (\Delta a)^{\overline{\lambda}} \end{pmatrix} \cdot \mathbf{M} \cdot \begin{pmatrix} (\Delta a)^{\overline{\lambda}} & 0 \\ 0 & (\Delta a)^{\lambda} \end{pmatrix}$$

while formulae (18) remains the same. The complex coefficients a_j of the outer decomposition now turn into

$$a_1(\Delta a) = K(\Delta a)^{\Lambda+\overline{\Lambda}} M_1 + \overline{K}(\Delta a)^{2\Lambda} M_2 + \ldots, \qquad a_2(\Delta a) = \overline{a_1(\Delta a)}. \tag{21}$$

Hence the energy release $\Delta\mathbf{U}$ while the crack tip moves from the tip $(-\Delta a, 0)$ to $(0,0)$ becomes

$$\Delta\mathbf{U} = (\Delta a)^{2\operatorname{Re}\lambda}\left(M_1|K|^2 + \operatorname{Re}(K^2\overline{M}_2 e^{-2i\ln(h)\operatorname{Im}\lambda})\right) + \ldots \tag{22}$$

Case 3: A real double eigenvalue λ ***with geometric multiplicity one***
While rewriting the inner decomposition of $u^{\Delta a}$ in terms of x-coordinates, one has to take into account the logarithmic terms in the power-law solutions. We introduce the matrix

$$\mathbf{Q}_{\pm} = \mathbf{Q}_{\pm}(\Delta a) = \begin{pmatrix} 1 & 0 \\ \pm\ln(\Delta a) & 1 \end{pmatrix}, \text{ then } \mathbf{Q}_{\pm}^{-1} = \mathbf{Q}_{\mp}.$$

Equalizing the coefficients in front of X^j and Y^j in the expansions eq. (17) and (12) now leads to the system

$$\begin{aligned}\mathbf{K} + \mathbf{m} \cdot \mathbf{a} &= \mathbf{Q}_-^\top \cdot (\Delta a)^{-\lambda} \mathbf{b}, \\ \mathbf{a} &= \mathbf{Q}_+ \cdot \mathbf{M} \cdot (\Delta a)^{\lambda} \mathbf{b}.\end{aligned}$$

From here we get

$$\begin{aligned}\mathbf{a} &= (\Delta a)^{2\lambda} \mathbf{Q}_+ \mathbf{M} \cdot \left(\mathbb{I} - \mathbf{Q}_+^\top \cdot \mathbf{m} \cdot \mathbf{Q}_+ \cdot (\Delta a)^{2\lambda} \mathbf{M} \right)^{-1} \cdot \mathbf{Q}_+^\top \cdot \mathbf{K} \\ &= (\Delta a)^{2\lambda} \mathbf{Q}_+ \cdot \mathbf{M} \cdot \mathbf{Q}_+^\top \cdot \mathbf{K} + \dots .\end{aligned}$$

Using (15) once more leads now to

$$\begin{aligned}2\Delta \mathbf{U} &= (\Delta a)^{2\lambda} \mathbf{K}^\top \cdot \mathbf{Q}_+ \cdot \mathbf{M} \cdot \mathbf{Q}_+^\top \cdot \mathbf{K} + \dots \\ &= (\Delta a)^{2\lambda} \Big(M_{11} \left(K_1 + K_2 \ln(\Delta a) \right)^2 + \\ &\qquad 2M_{12} K_2 \left(K_1 + K_2 \ln(\Delta a) \right) + M_{22} K_2^2 \Big) + \dots\end{aligned}$$

Note that all these formulae remain valid but with an additional factor -1, if we start with the crack tip already on the interface. Thereby the following obvious changes must be taken into account: the weight functions η^j are now solutions to the interface problem in the plane with half infinite crack where the tip is situated in $(0, 1)$, the matrix $\mathbf{M}$ is positive definite, and formula $(11)_2$ now reads $\Delta \mathbf{U} = -\frac{1}{2} \int_\Gamma p \cdot (u^{\Delta a} - u^0)\, ds$.

CONCLUSIONS FOR THE CRACK GROWTH

While for a selvage crack in a homogenous material the energy release rate $\mathcal{G}$ always satisfies

$$\mathcal{G} =: \lim_{h \downarrow 0} \frac{-\Delta U}{h} = \mathbf{K}^\top \cdot \mathbf{M} \cdot \mathbf{K} > 0,$$

it may happen here that $\mathcal{G} = 0$ or $\mathcal{G} = \infty$. This means the crack cannot reach the interface if

$$\lambda_1 > 1/2, \text{ (Case 1) }, \operatorname{Re} \lambda > 1/2 \text{ (Case 2) }, \Lambda > 1/2 \text{ (Case 3) },$$

because the $\mathcal{G} = 0$ and Condition (4) cannot be met. If the relation $>$ is replaced by $<$ we have $\mathcal{G} = \infty$ hence the crack undergoes at least a phase of unstable propagation.

It may as well happen that there is an equality in the relation above. In case 1 the energy release rate is determined by the first summand in eq. (19) alone, if $\lambda_1 < \lambda_2$. The crack tip will move in direction of the interface if $\mathcal{G}$ overcomes the critical energy release rate in the first material. In case 3 we have $\mathcal{G} = \infty$ unless $K_2 = 0$, then again K_1 has to be critical. In case 2 there appears an oscillating term in formula (22), however we may conclude that the crack can reach the interface if $|M_2|$ is sufficiently small in comparison to $|M_1|$.

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